Application of Gauss's Law

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Divergence of Electric Field

Now we shall calculate divergence of $\vec{E}$ directly-

We know that

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\text{over all space}} \rho(\vec{r}) \frac{\hat{R}}{R^2} dv'$$

$$\text{where } \vec{R} = \vec{r} - \vec{r}'$$

- Originally the above integration was over the volume occupied by the charge, but one can extend it to all space.
  - ✓ because in the exterior region $\rho = 0$

So

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0} \int \nabla \cdot \left(\frac{\hat{R}}{R^2} \right) \rho(\vec{r}') dv'$$

We know that $$\nabla \cdot \left(\frac{\hat{R}}{R^2} \right) = 4\pi \delta^3 \left(\vec{R} \right)$$

$$\therefore \quad \nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0} \int 4\pi\delta^3 \left(\vec{r} - \vec{r}' \right) \rho \left(\vec{r}' \right) dv'$$

$$= \frac{\rho\left(\vec{r}\right)}{\varepsilon_0}$$

Gauss's law in differential form

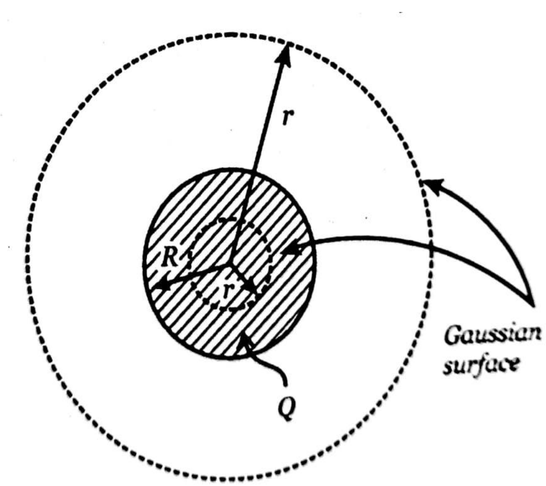
Again $$\oint_S \vec{E} \cdot d\vec{s} = \int_v \nabla \cdot \vec{E} \, dv$$

$$= \frac{1}{\varepsilon_0} \int_v \rho dv$$

$$= \frac{Q_{enc}}{\varepsilon_0}$$ ***Gauss's law in integral form***

Application of Gauss's Law

Example 1: Find the electric field outside and inside a uniformly charged solid sphere of radius R and total charge q.



For outside the sphere ($r > R$):

✓ Let us draw a concentric spherical Gaussian Surface of radius $r > R$

✓ The point at which the field is to be calculated lies on this Gaussian surface.

✓From the spherical symmetry it is clear that the direction of Electric field at every point on the Gaussian Surface is radially outward and the magnitude of the electric filed is same at every point on the Gaussian surface.

According to Gauss's law

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{q_{en}}{\varepsilon_0}$$

$$q_{en} = q$$

$$\oint_S \vec{E} \cdot d\vec{s} = \oint_S E ds$$

$$\oint_S E ds = E \oint_S ds = E.4\pi r^2$$

$$E.4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$\text{or } E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

$$\text{so, } \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

✓It should be noted that the field outside the sphere is exactly the same as it would have been if all the charges had been concentrated at the centre

For inside the sphere ($r < R$):

✓ Now draw a concentric spherical Gaussian Surface of radius $r < R$

According to Gauss's law $$\oint_S \vec{E} \cdot d\vec{s} = \frac{q_{en}}{\varepsilon_0}$$

Here q_{en} is the charge enclosed by the Gaussian surface

Volume charge density ρ

$$q_{en} = \oint_V \rho dv$$

$$= \rho \oint_V dv \;\; [\text{Since } \rho \text{ is uniform}]$$

$$= \rho . \frac{4\pi r^3}{3}$$

$$\oint_S \vec{E} \cdot d\vec{s} = \oint_S E\, ds$$

$$\oint_S Eds = E\oint_S ds = E.4\pi r^2$$

$$E.4\pi r^2 = \frac{q_{en}}{\varepsilon_0}$$

$$\text{or} \quad E = \frac{1}{4\pi\varepsilon_0} \frac{\rho.\frac{4\pi r^3}{3}}{r^2}$$

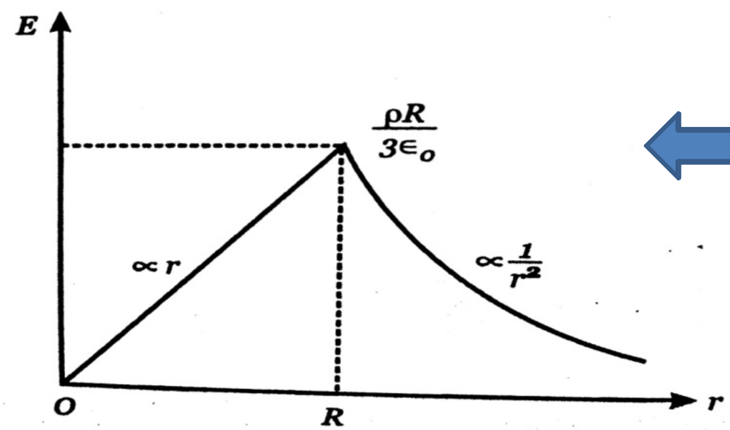
$$= \frac{\rho r}{3\varepsilon_0}$$

$$\therefore \qquad \vec{E} = \frac{\rho \ \vec{r}}{3 \ \varepsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}\vec{r}$$

At the surface of the sphere ($r = R$):

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{r} = \frac{\rho R \hat{r}}{3\varepsilon_0}$$



The variation of field with the distance from the centre of the sphere

Important note regarding application of Gauss's Law

- Gauss's law is always true but not always useful

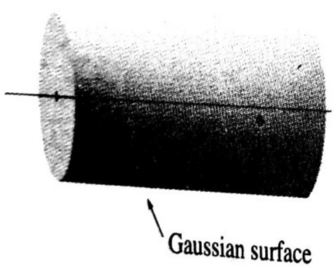
- If ρ had been at any arbitrary rate i.e. not symmetric and chosen Gaussian surface had been of any arbitrary shape then it would have been **true** that flux of $\vec{E}$ is $\frac{q_{enc}}{\varepsilon_0}$.

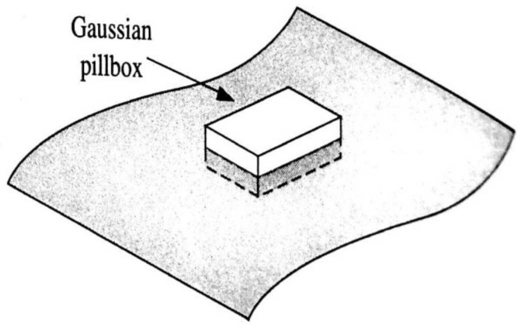
- **But** it would not have been certain that $\vec{E}$ was in the same direction as $d\vec{s}$ and constant in magnitude over the surface, and without this we could not pulled E out of the integral.

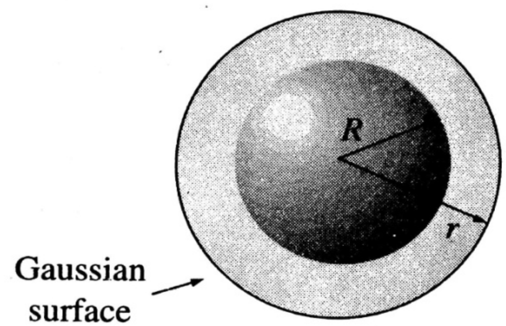
- **So, symmetry plays crucial role to this application of Gauss's Law**

Symmetry

Symmetry	Gaussian Surface
Spherical	Concentric sphere
Cylindrical	Coaxial Cylinder
Plane	Pillbox which straddles the surface







✓ Cylindrical symmetry requires infinitely long cylinder

✓ Plane symmetry requires planes extending to infinite in all directions.

✓ **But we use long cylinders or large place surface**

Example 2

A long cylinder carries a charge density that is proportional to the distance from the axis: $\rho = ks$, where k is a constant. Find the electric field inside and outside the cylinder. [radius of the cylinder a]

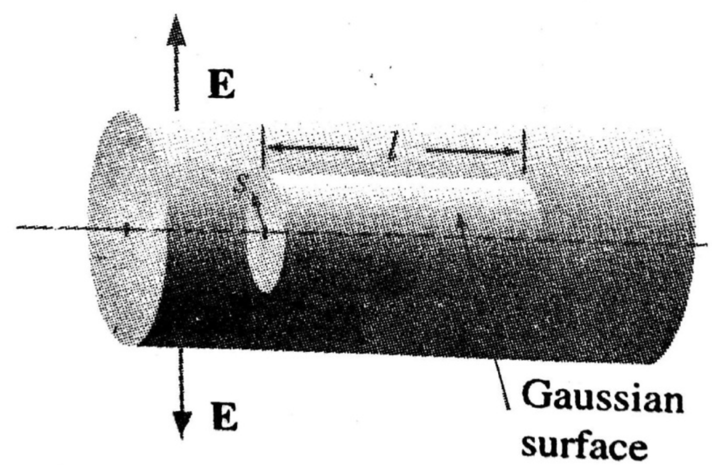
Solution

Inside the cylinder $s < a$:

Gaussian surface: Co-axial cylinder of length l and radius s .

According to Gauss's law

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{q_{en}}{\varepsilon_0}$$



$$q_{enc} = \int \rho dv$$

$$= \int k s' \left(s' d s' d\phi \, dz \right)$$

$$= 2\pi k l \int_0^s s'^2 ds'$$

$$= \frac{2}{3} \pi k l s^3$$

From symmetry it is clear that $\vec{E}$ must point radially outward and have equal magnitude of E over the curved Gaussian surface.

The two end surface contribute nothing to the flux as $\vec{E}$ is perpendicular to $d\vec{s}$.

$$\oint_S E ds = E \oint_S ds = E.2\pi sl$$

$$\therefore \quad E.2\pi sl = \frac{q_{en}}{\varepsilon_0}$$

$$\therefore \quad E.2\pi sl = \frac{1}{\varepsilon_0}\frac{2}{3}\pi kls^3$$

$$\vec{E} = \frac{1}{3\varepsilon_0} ks^2 \hat{s}$$

outside the cylinder $s > a$:

$$q_{enc} = \int \rho \, dv$$

$$= 2\pi k l \int_0^a s'^2 \, ds'$$

$$= \frac{2}{3} \pi k l a^3$$

From symmetry it is clear that $\vec{E}$ must point radially outward and have equal magnitude of E over the curved Gaussian surface.

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$$\therefore \quad E.2\pi sl = \frac{1}{\varepsilon_0}\frac{2}{3}\pi kla^3$$

$$\vec{E} = \frac{k}{3\varepsilon_0}\frac{a^3}{s}\hat{s}$$

Example 3

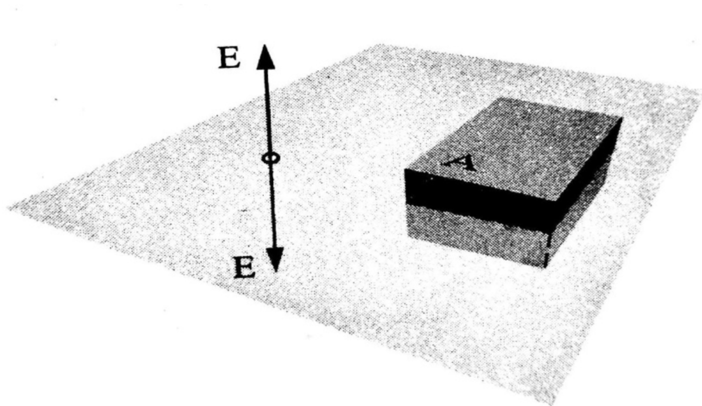
An infinite plane carries a uniform surface charge σ . Find its electric field .

Solution

Gaussian surface: Pillbox extending equal distance above and below the plane. Let A is the area of the lid of the pillbox.

According to Gauss's law

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{q_{en}}{\varepsilon_0}$$



Here $q_{enc} = \sigma A$

➢From symmetry it is clear that $\vec{E}$ points away from the plane.

➢ The side surfaces contribute nothing to the flux as $\vec{E}$ is perpendicular to $d\vec{s}$.

$$\oint_S Eds = 2EA$$

$$So \qquad 2EA = \frac{q_{en}}{\varepsilon_0}$$

$$\therefore \quad \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

$\hat{n}$ is unit vector pointing away from the surface

Field of an infinite plane is independent of distance from the surface.

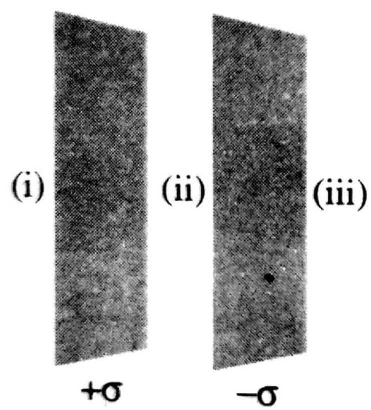
What about Coulomb's law $E \sim \dfrac{1}{r^2}$??

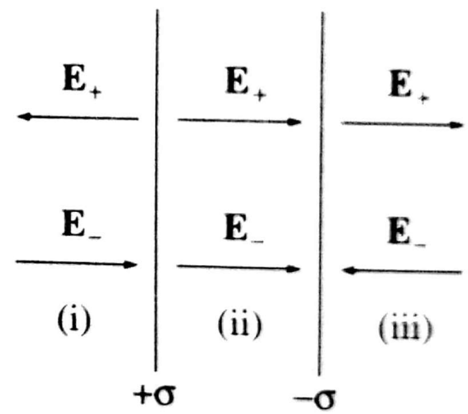
Object	Variation of Electric field
Electric field of a Sphere	$E \sim \frac{1}{r^2}$
Electric field of an infinite line	$E \sim \frac{1}{r}$
Electric field of an infinite plane	$E = \text{constant}$

Example 3

Two infinite parallel planes carry equal but opposite uniform surface charge $\pm\sigma$. Find the electric field in i) to the left of both ii) between them iii) to the right of the both.

Solution



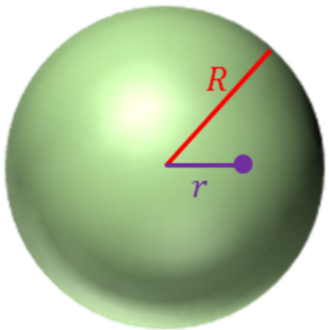


Region (i) and (iii) : $\vec{E} = \vec{0}$

Region (ii): $\vec{E} = \dfrac{\sigma}{\varepsilon_0}\hat{i}$

Problem

Find the electric field inside a sphere which carries a charge density proportional to the distance from the origin, $\rho = kr$, k is a constant.



$$\oint_S \vec{E} \cdot d\vec{s} = \frac{q_{en}}{\varepsilon_0}$$

$$q_{enc} = \int \rho \, dv$$

$$= \int kr' 4\pi r'^2 dr'$$

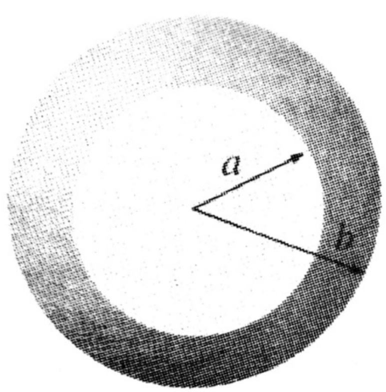
$$= \frac{\pi k}{\varepsilon_0} r^4$$

$$E.4\pi r^2 = \frac{q_{en}}{\varepsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\pi k r^2 \hat{r}$$

Problem

Hollow spherical shell carries charge density ,$\rho = \frac{k}{r^2}$, k is a constant. In the region $a \le r \le b$. Find the electric field in three regions i) $r < a$ ii) $a < r < b$ iii) $r > b$. Plot E as a function of r .



$$\oint_S \vec{E} \cdot d\vec{s} = \frac{q_{en}}{\varepsilon_0}$$

i) Region $r < a$

$$q_{enc} = 0$$

$$\vec{E} = \vec{0}$$

ii) Region $a < r < b$

$$q_{enc} = \int \rho \, dv$$

$$= \int_a^r \frac{k}{r'^2} 4\pi r'^2 dr'$$

$$= 4\pi k \left(r - a \right)$$

$$\therefore \quad \vec{E} = \frac{k}{\varepsilon_0} \left(\frac{r-a}{r^2} \right) \hat{r}$$

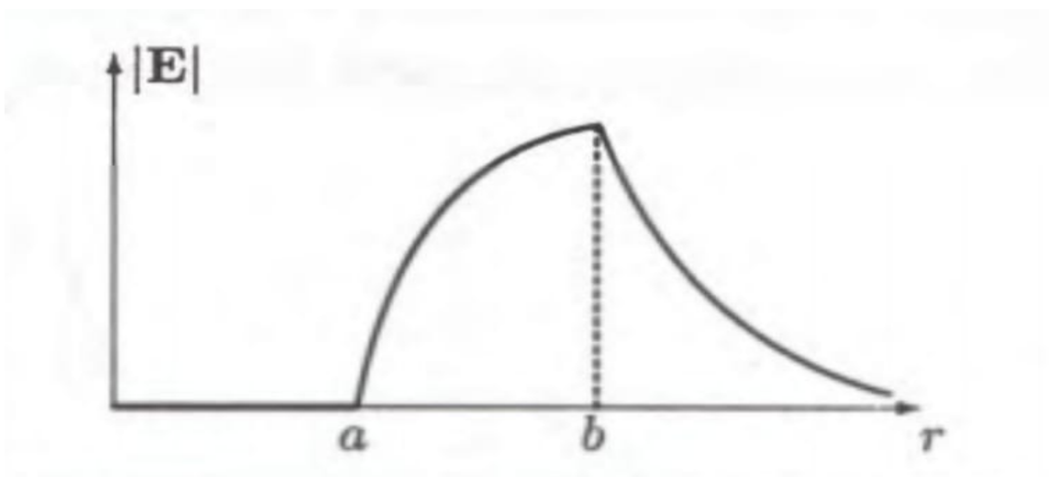
iii) Region $r > b$

$$q_{enc} = \int \rho \, dv$$

$$= \int_a^b \frac{k}{r'^2} 4\pi r'^2 dr'$$

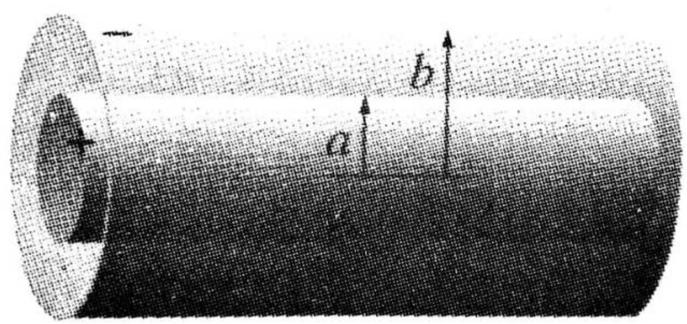
$$= 4\pi k \left(b - a\right)$$

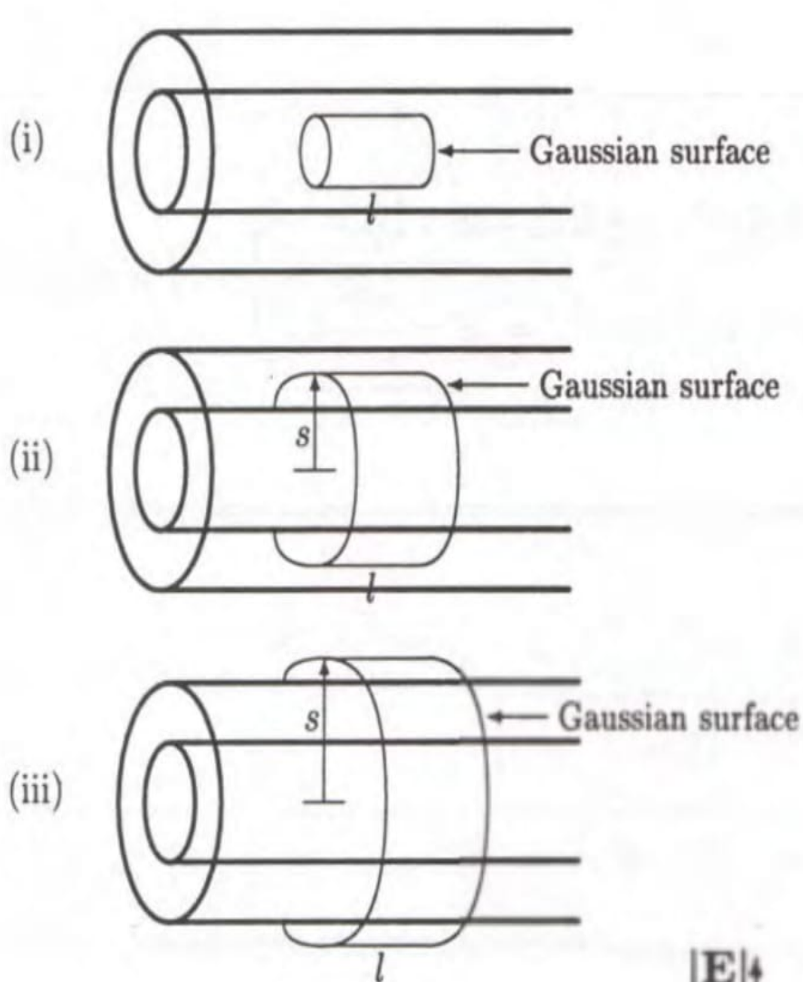
$$\therefore \quad \vec{E} = \frac{k}{\varepsilon_0}\left(\frac{b-a}{r^2}\right)\hat{r}$$



Problem

A long co-axial cable carries a uniform volume charge density ρ on the inner cylinder of radius a and a uniform surface charge density σ on the outer cylindrical shell of radius b. The surface charge is negative and of just the right magnitude so that the cable as a whole is electrically neutral. Find the electric field in each of the three regions: i) inside the inner cylinder $s < a$, ii) between the cylinders $a < s < b$ iii) outside the cable $s > b$. Plot $|\vec{E}|$ as a function of s.





$$\oint_S \vec{E}.d\vec{s} = \frac{q_{enc}}{\varepsilon_0} \qquad \Rightarrow E.2\pi sl = \frac{\rho\pi s^2 l}{\varepsilon_0}$$

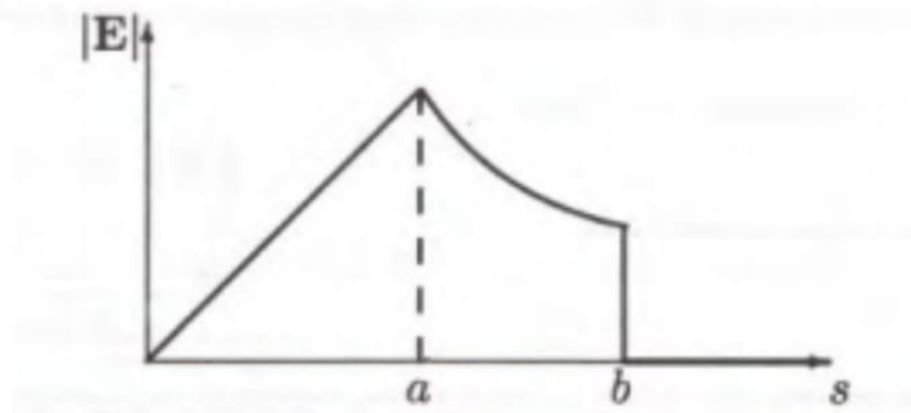
$$\therefore \quad \vec{E} = \frac{\rho\, s}{2\,\varepsilon_0}\,\hat{s}$$

$$\oint_S \vec{E}.d\vec{s} = \frac{q_{enc}}{\varepsilon_0} \qquad \Rightarrow E.2\pi sl = \frac{\rho\pi a^2 l}{\varepsilon_0}$$

$$\therefore \quad \vec{E} = \frac{\rho\, a^2}{2\,\varepsilon_0\, s}\,\hat{s}$$

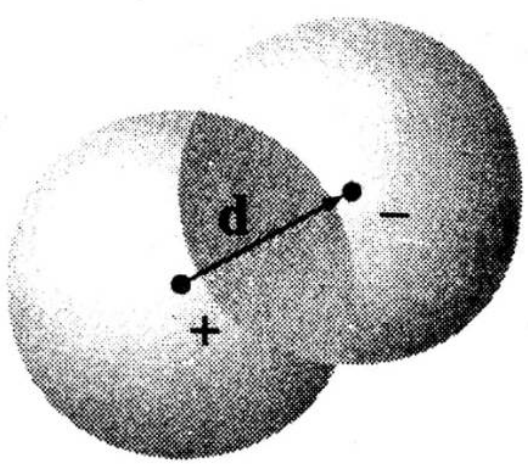
$$\oint_S \vec{E}.d\vec{s} = \frac{q_{enc}}{\varepsilon_0} \qquad \Rightarrow E.2\pi sl = \frac{0}{\varepsilon_0}$$

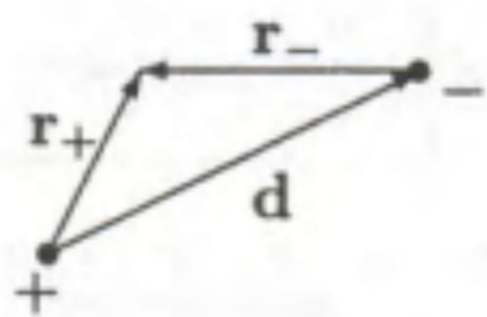
$$\vec{E} = \vec{0}$$



Problem

Two spheres, each of radius R and carrying uniform charge densities $+\sigma$ and $-\sigma$, respectively, are placed so that they partially overlap. Let the vector from the positive center to the negative center $\vec{d}$. Show that the field in the region of overlap is constant and find its value.





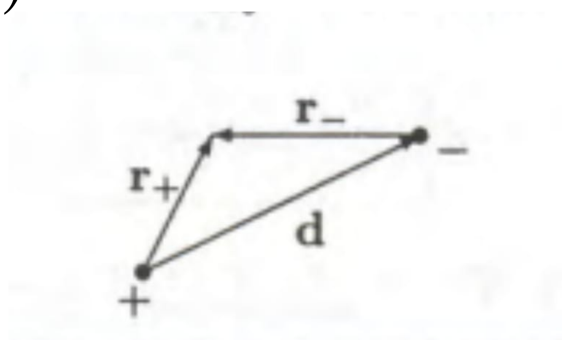
Field inside the positive sphere $\vec{E}_+ = \frac{\rho}{3\varepsilon_0}\vec{r}_+$

Where $\vec{r}_+$ is the vector from the positive charged centre to the point in question.

Similarly, field inside the negative sphere $\vec{E}_- = \frac{\rho}{3\varepsilon_0}\vec{r}_-$

Where $\vec{r}_-$ is the vector from the positive charged centre to the point in question.

So, the total field is $\vec{E} = \frac{\rho}{3\varepsilon_0}\left(\vec{r}_+ - \vec{r}_-\right)$

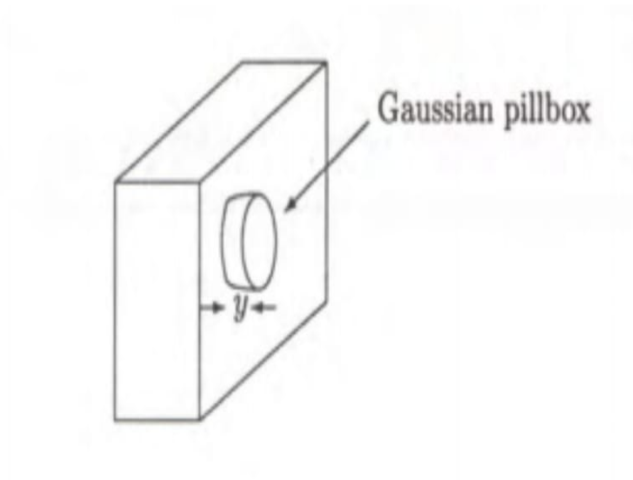


But from the figure $\vec{r}_+ - \vec{r}_- = \vec{d}$

$$\therefore \quad \vec{E} = \frac{\rho}{3\varepsilon_0}\vec{d}$$

Problem

An infinite plane slab, of thickness $2d$, carries a uniform volume charge density ρ. Find the electric field, as a function of y, where $y = 0$ at the center. Plot $|\vec{E}|$ versus y, calling $|\vec{E}|$ positive when it points in the $+y$ direction and negative when it points in the $-y$ direction.



Face area of the pillbox is A

Symmetry:

- Electric field $\vec{E} = \vec{0}$ on the xz plane
- Points in the $+y$ and $-y$ directions

Gaussian Surface

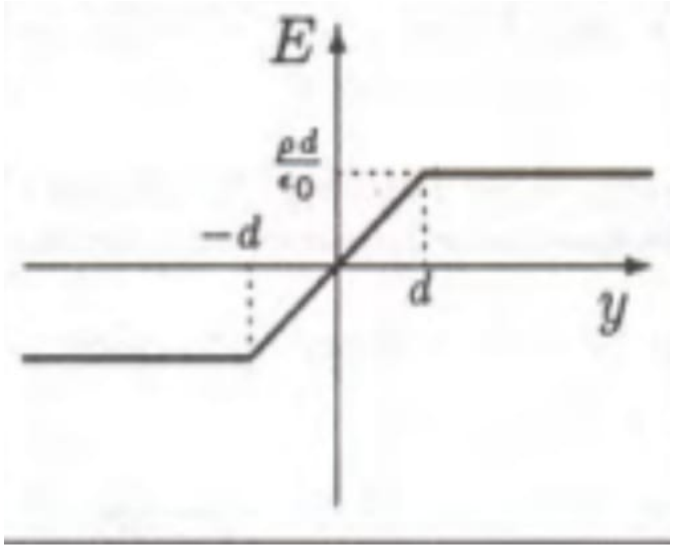
A pillbox with one face in the xz plane and the other face is at y.

For $|y| < d \ (i.e.\ y > -d \ \& \ y < d\ )$:

$$\oint_S \vec{E}.d\vec{s} = \frac{q_{enc}}{\varepsilon_0}$$

$$EA = \frac{\rho A y}{\varepsilon_0}$$

$$\vec{E} = \frac{\rho y}{\varepsilon_0}\hat{y}$$



For $|y| > d \ (i.e.\ y < -d \ \& \ y > d\ )$:

$$\oint_S \vec{E}.d\vec{s} = \frac{q_{enc}}{\varepsilon_0} \quad \Rightarrow EA = \frac{\rho A d}{\varepsilon_0}$$

$$\vec{E} = \frac{\rho d}{\varepsilon_0}\hat{y}$$

Reference:

Introduction to Electrodynamics

David J. Griffiths