Dirac-delta function and different Limit representation of Dirac-delta function

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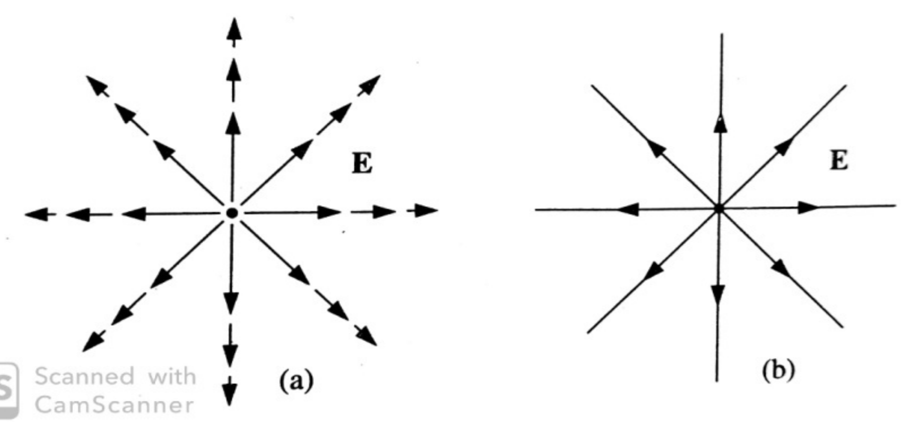
first calculate the divergence of $\dfrac{\hat{r}}{r^2}$

Let $$\vec{E} = \frac{\hat{r}}{r^2}$$



This vector function is directed radially outward and has a large positive divergence

What happens, if we want to calculate the divergence of $\vec{E}$?

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2} \right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(1)$$

$$= 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

???????

Surface integration of the vector function over a sphere of radius R , centered at the origin

$$\oint \vec{E} \cdot d\vec{S} = \int \left(\frac{\hat{r}}{R^2} \right) \cdot \left(R^2 \sin(\theta) d\theta d\phi \, \hat{r} \right)$$

$$= \int_0^{\pi} \sin(\theta) d\theta \int_0^{2\pi} d\phi$$

$$= 4\pi \qquad ....(2)$$

volume integral of the divergence of the vector function is

$$\int \nabla \cdot \vec{E} \, dv = 0 \qquad ....(3)$$

➢ According to Divergence theorem equation no. (2) and (3) must be same

???????

Source of the problem $r = 0$

At this point the vector function $\vec{E}$ blows up.

$$\nabla \cdot \vec{E} = 0 \quad \text{except at} \quad r = 0$$

The surface integral is independent of R. So the surface integral will be 4π for any sphere centered at the origin, whatever be the radius of the sphere.

Entire contribution of the surface integral comes from the point $r=0$.

For the consistency of the divergence theorem

$$\int \nabla \cdot \vec{E}\, dv = 0 \qquad \text{except } r = 0$$

$$= 4\pi \qquad \text{otherwise}$$

$\nabla \cdot \vec{E}$ has an unusual property

It vanishes everywhere except at one point and its integral over any volume containing that point is 4π.

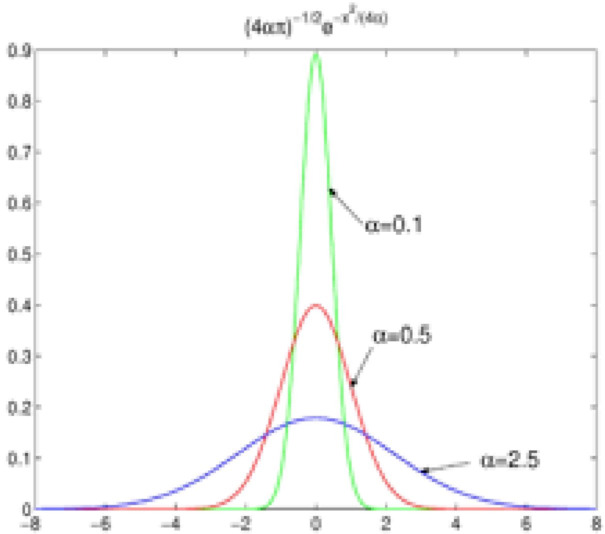
✓**This type of function is called Dirac Delta function.**

One dimensional Dirac Delta function:

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$



✓ **An important property of a Delta function:**

$$f(x)\delta(x) = f(0)\delta(x)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty}\delta(x)\,dx$$
$$= f(0)$$

- One can take the limits of integration from $-\varepsilon$ to $+\varepsilon$ instead of $-\infty$ to $+\infty$

- ✓ **if the peak of the delta function be at** $x = a$

$$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1$$

- ✓ **The product of this delta function with an arbitrary function** $f(x)$

$$f(x)\delta(x-a) = f(a)\delta(x)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$

Dirac Delta function in three dimension

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\int_{all\ space} \delta^3(\vec{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\,dxdydz = 1$$

And

$$\int_{all\,space} f(\vec{r})\,\delta^3(\vec{r}-\vec{a})\,d\tau = f(\vec{a})$$

Resolving the paradox introduced beginning of the talk

$$\nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi\delta^3(\vec{r})$$

It follows that

$$\nabla^2 \frac{1}{r} = -\nabla \cdot \frac{\hat{r}}{r^2} = -4\pi\delta^3(\vec{r})$$

Limiting representation of Dirac Delta function

- Technically $\delta(x)$ is not a function, since its value is not finite at $r = 0$

- In mathematical literature it is known as a generalized function or distribution

- Dirac delta function basically is the limit of sequence of functions

- There are many limit representation of Dirac delta functions

Examples:…………….

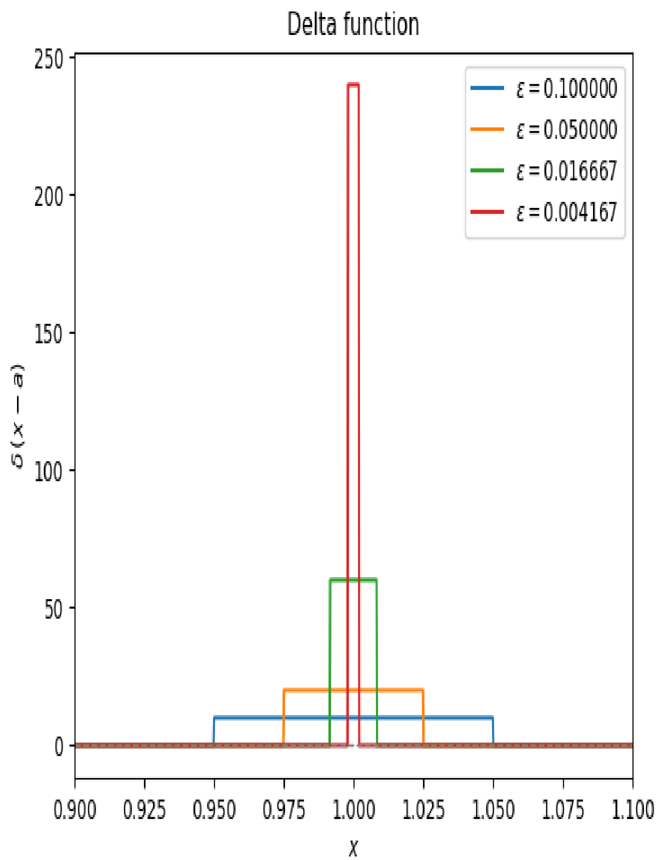
Limit of sequence of Rectangles $R_\varepsilon(x)$

$$\delta(x) = \lim_{\varepsilon \to 0} R_\varepsilon(x)$$

$$\text{where,} \quad R_\varepsilon(x) = \begin{cases} 0 & \text{for } x < -\dfrac{\varepsilon}{2} \\ \dfrac{1}{\varepsilon} & \text{for } -\dfrac{\varepsilon}{2} < x < \dfrac{\varepsilon}{2} \\ 0 & \text{for } x > \dfrac{\varepsilon}{2} \end{cases}$$

Width of rectangles ε

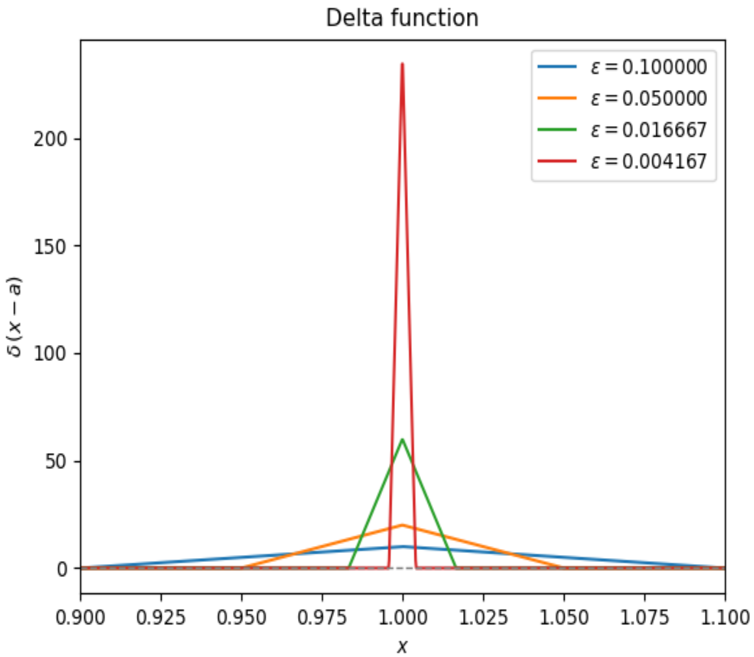
height of rectangles $\dfrac{1}{\varepsilon}$



Limit of sequence of isosceles triangles $T_\varepsilon(x)$:

$$\delta(x) = \lim_{\varepsilon \to 0} T_\varepsilon(x)$$

where, $$T_\varepsilon(x) = \begin{cases} 0 & \text{for } |x| > \varepsilon \\ \dfrac{\left(1 - \dfrac{|x|}{\varepsilon}\right)}{\varepsilon} & \text{for } |x| < \varepsilon \end{cases}$$



Here the base of the triangle $T_\varepsilon(x)$ is 2ε

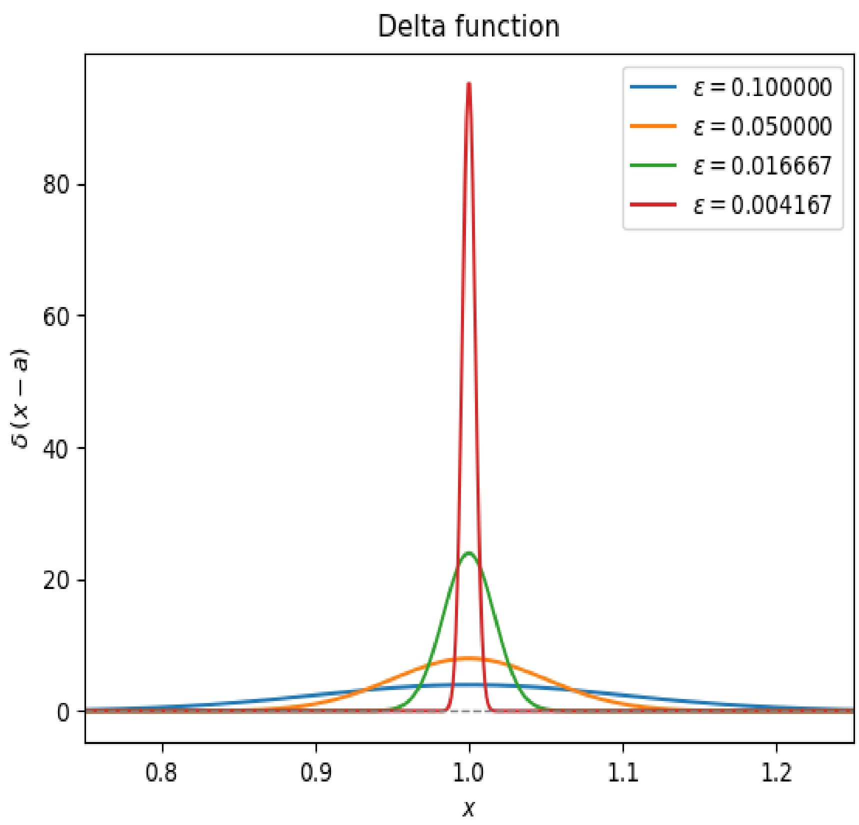
and height $\dfrac{1}{\varepsilon}$

The area of the triangle is unity

Limit of sequence of Gaussian functions $G_\varepsilon(x)$

$$\delta(x) = \lim_{\varepsilon \to 0} G_\varepsilon(x)$$

$$\text{where,} \qquad G_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}} e^{-\frac{x^2}{\varepsilon^2}}$$



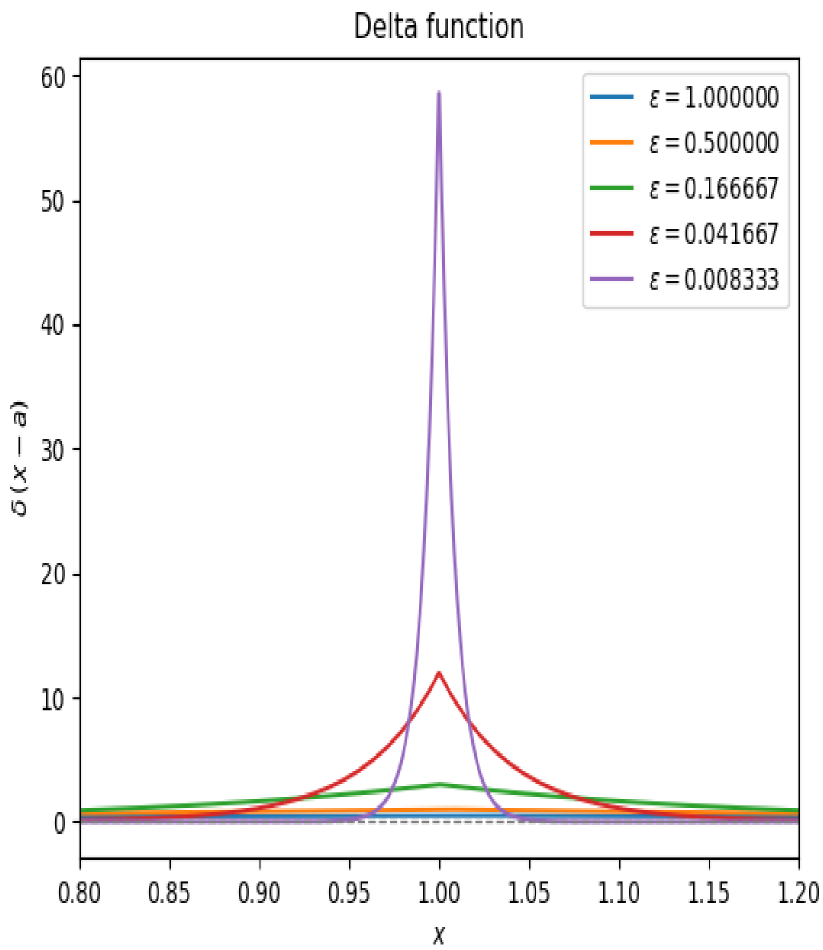
This is the normalized Gaussian distribution function. The area under the curve is unity and the peak value $\dfrac{1}{\varepsilon\sqrt{\pi}}$

Limit of sequence of Exponential functions $E_\varepsilon(x)$

$$\delta(x) = \lim_{\varepsilon \to 0} E_\varepsilon(x)$$

$$\text{where,} \qquad E_\varepsilon(x) = \frac{1}{2\varepsilon} e^{-\frac{|x|}{\varepsilon}}$$

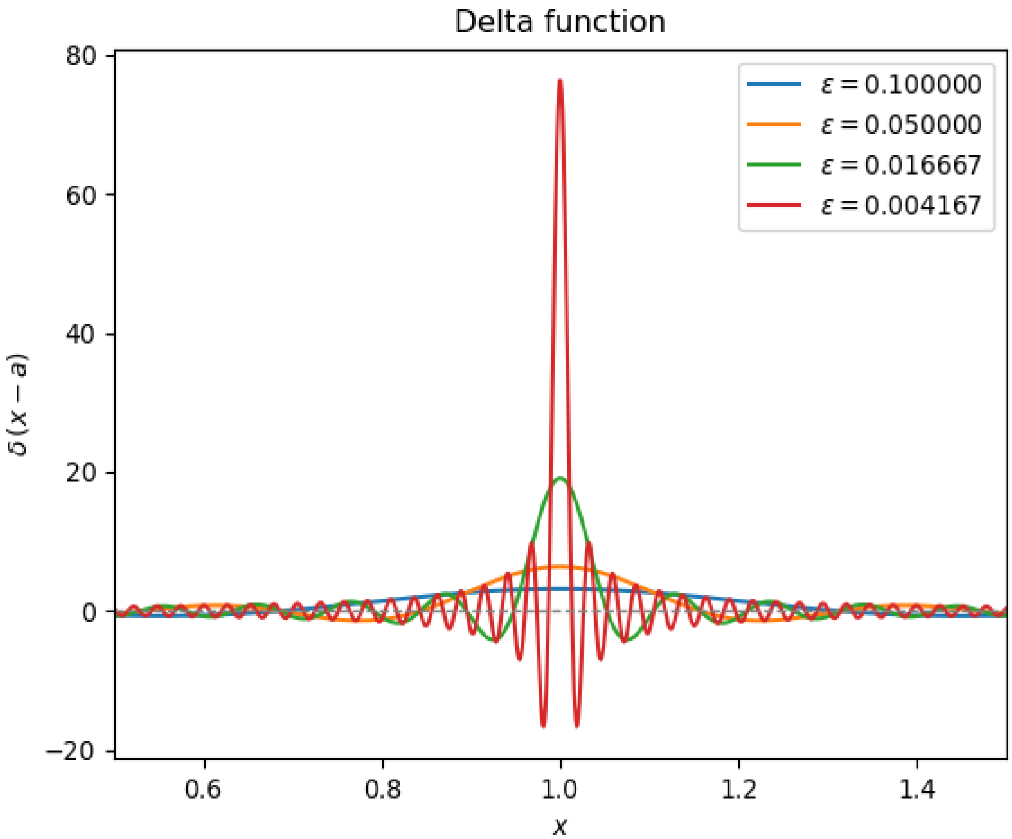
The area under the curve is unity and the peak value $\frac{1}{2\varepsilon}$



Limit of sequence of sinc functions $S_\varepsilon(x)$

$$\delta(x) = \lim_{\varepsilon \to 0} S_\varepsilon(x)$$

where, $$S_\varepsilon(x) = \frac{\sin(x/\varepsilon)}{\pi x}$$



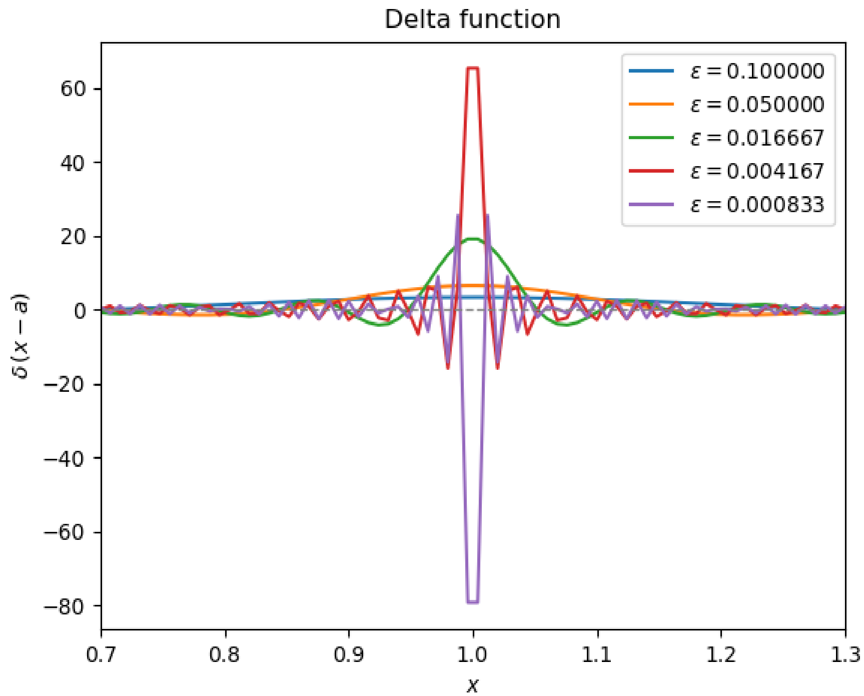
This function arises frequently in signal processing and the theory of Fourier transforms. The full name of this function is since cardinal but it is commonly referred to "sinc" function.

Limit of sequence of modified sinc function $Sm_\varepsilon(x)$

$$\delta(x) = \lim_{\varepsilon \to 0} Sm_\varepsilon(x)$$

$$\text{where,} \quad Sm_\varepsilon(x) = \frac{1}{2\pi} \frac{\sin\left[\frac{\left(1+\frac{2}{\varepsilon}\right)x}{2}\right]}{\sin\left(\frac{x}{2}\right)}$$

The peak value $\left(\frac{1}{2}+\frac{1}{\varepsilon}\right)/\pi$

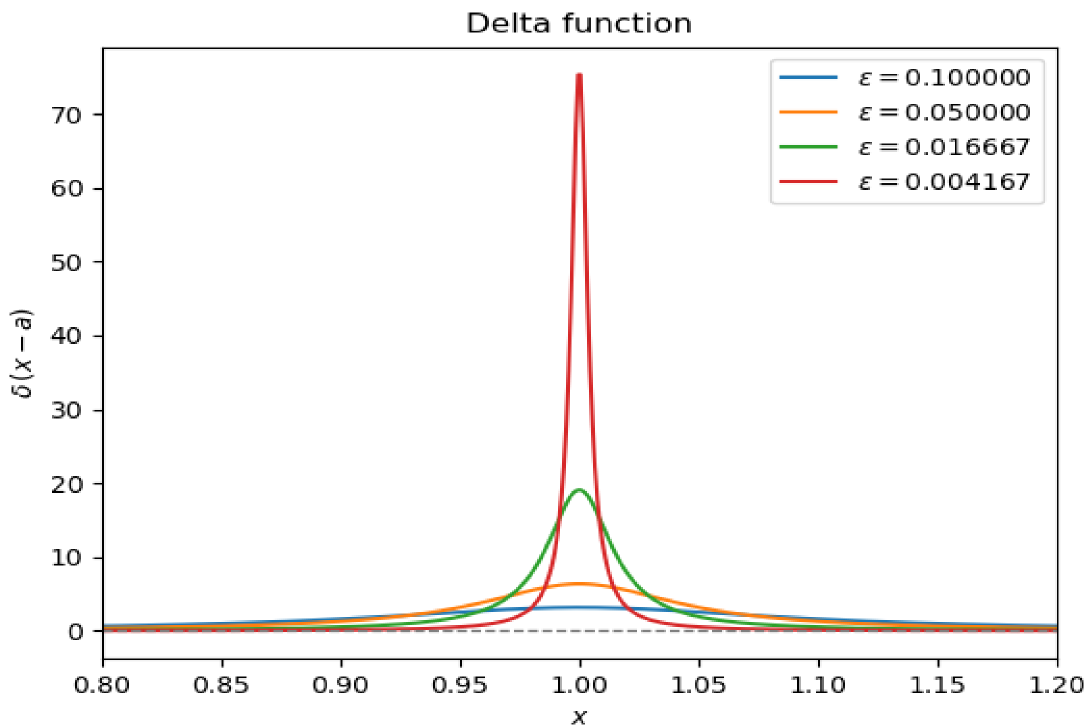


Limit of sequence of Lorentzian $L_\varepsilon(x)$

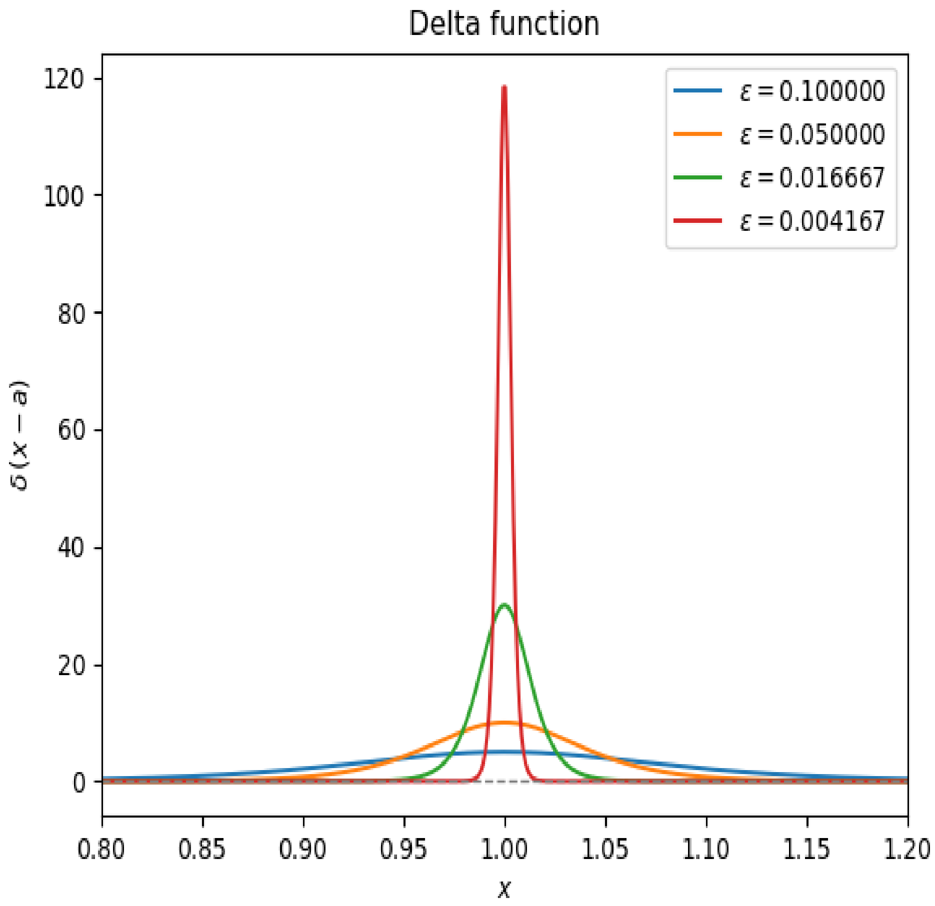
$$\delta(x) = \lim_{\varepsilon \to 0} L_\varepsilon(x)$$

The peak value $\dfrac{1}{\pi\varepsilon}$

where, $$L_\varepsilon(x) = \frac{1}{\pi\varepsilon} \frac{1}{1+\dfrac{x^2}{\varepsilon^2}}$$



Limit of sequence of Inverse Cosh Square functions $C_\varepsilon(x)$



$$\delta(x) = \lim_{\varepsilon \to 0} C_\varepsilon(x)$$

where, $$C_\varepsilon(x) = \frac{1}{2\varepsilon} \frac{1}{\cosh^2\left(\frac{x}{\varepsilon}\right)}$$

The peak value $\frac{1}{2\varepsilon}$

Note: Though δ is not a legitimate function but the integrals over δ are perfectly acceptable.

If $\displaystyle\int_{-\infty}^{\infty} f(x)\,d_1(x)\,dx = \int_{-\infty}^{\infty} f(x)\,d_2(x)\,dx$ for all function $f(x)$

then $d_1(x) = d_2(x)$

Example

Show that $\delta(kx) = \frac{1}{|k|}\delta(x)$ Where k is nonzero constant

Answer

Consider an arbitrary function $f(x)$ and consider the following integral

$$\int_{-\infty}^{\infty} f(x)\delta(kx)\,dx$$

Change of variable $y = kx$

$$dx = \frac{1}{k}dy$$

$$\int_{-\infty}^{\infty} f(x)\delta(kx)\,dx = \pm\frac{1}{k}\int_{-\infty}^{\infty} f\left(\frac{y}{k}\right)\delta(y)\,dy$$

+ve sign for $k > 0$

-ve sign for $k < 0$

$$= \pm \frac{1}{k} f(0)$$

$$= \frac{1}{|k|} f(0)$$

$$= \frac{1}{|k|} \int_{-\infty}^{\infty} f(x)\delta(x)\,dx$$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\delta(kx)\,dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right]dx$$

Hence $\delta(kx) = \frac{1}{|k|}\delta(x)$

Example

Show that $$x\frac{d}{dx}\left(\delta\left(x\right)\right)=-\delta\left(x\right)$$

Solution

$$\int_{-\infty}^{\infty} f\left(x\right)\left[x\frac{d}{dx}\left(\delta\left(x\right)\right)\right]dx$$

$$= xf\left(x\right)\delta\left(x\right)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}\left(xf\left(x\right)\right)\delta\left(x\right)dx$$

$$= 0 - \int_{-\infty}^{\infty} x\frac{df}{dx}\delta\left(x\right)dx - \int_{-\infty}^{\infty} f\left(x\right)\delta\left(x\right)dx$$

$$= 0 - 0 - f\left(0\right)$$

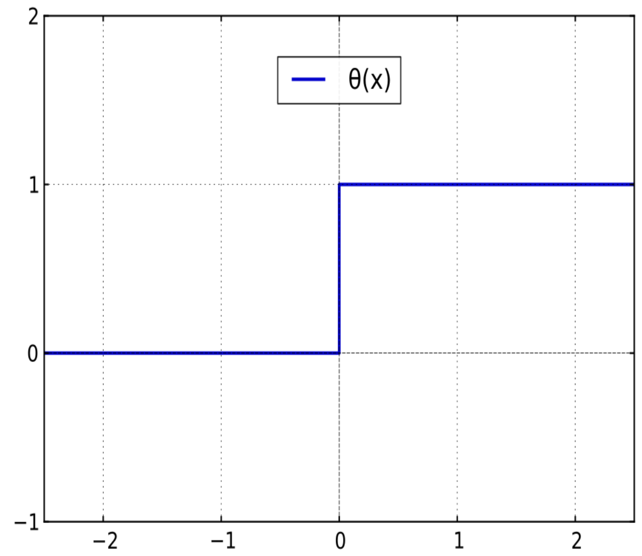
$$= -\int_{-\infty}^{\infty} f(x)\delta(x)\,dx$$

$$\therefore \quad x\frac{d}{dx}\left(\delta(x)\right) = -\delta(x)$$

Example

If $\theta(x)$ be a step function:

$$\theta(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & if\ x \leq 0 \end{cases}$$



$$\text{Show that } \frac{d\theta}{dx} = \delta(x)$$

Solution

$$\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx$$

$$= f(x)\theta(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} \frac{df}{dx}\theta(x)dx$$

$$= f(\infty) - \int_{0}^{\infty} \frac{df}{dx}\theta(x)dx$$

$$= f(\infty) - \big(f(\infty) - f(0)\big)$$

$$= f(0)$$

$$= \int_{-\infty}^{\infty} f(x)\delta(x)dx$$

$$\therefore \quad \frac{d\theta}{dx} = \delta(x)$$

Problem-1

$$\int_{2}^{6}\left(3x^{2}-2x-1\right)\delta\left(x-3\right)dx=?$$

Solution

$$\int_{2}^{6}\left(3x^{2}-2x-1\right)\delta\left(x-3\right)dx$$

$$=\int_{2}^{6} f\left(x\right)\delta\left(x-3\right)dx$$

$$f\left(x\right)=\left(3x^{2}-2x-1\right)$$

$$=f\left(3\right)$$

$$=20$$

Problem-2

$$\int_{-2}^{2}\left(2x+3\right)\delta\left(3x\right)dx$$

Solution

$$=\int_{-2}^{2} f\left(x\right)\frac{1}{3}\delta\left(x\right)dx$$

$$f\left(x\right)=\left(2x+3\right)$$

$$= \frac{1}{3} f(0)$$

$$= 1$$

Problem-3

$$\int_{-\infty}^{a} \delta(x-b)\,dx = ?$$

Solution

$$\text{for } a > b: \qquad \int_{-\infty}^{a} \delta(x-b)\,dx = 1$$

$$\text{for } a < b: \qquad \int_{-\infty}^{a} \delta(x-b)\,dx = 0$$

Reference:

Introduction to Electrodynamics

David J. Griffiths