Electric Potential

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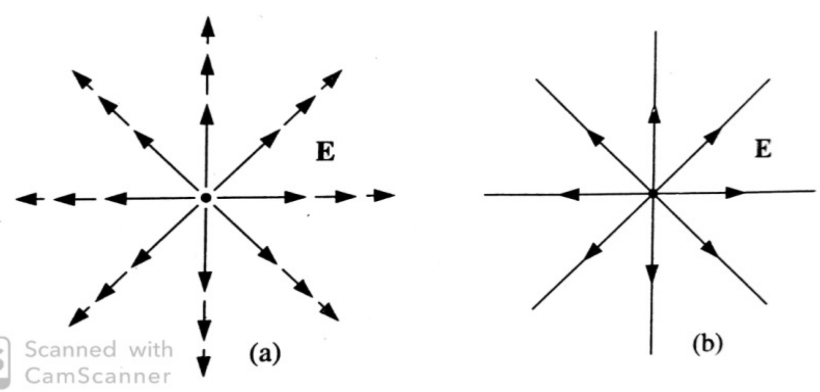
The Curl of $\vec{E}$

- **Before we going to calculate the Curl of $\vec{E}$ directly, we shall discuss it qualitatively first.**

- **Consider a simplest charge distribution, a single point charge located at the origin.**

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\hat{r}$$

Representation of this field graphically

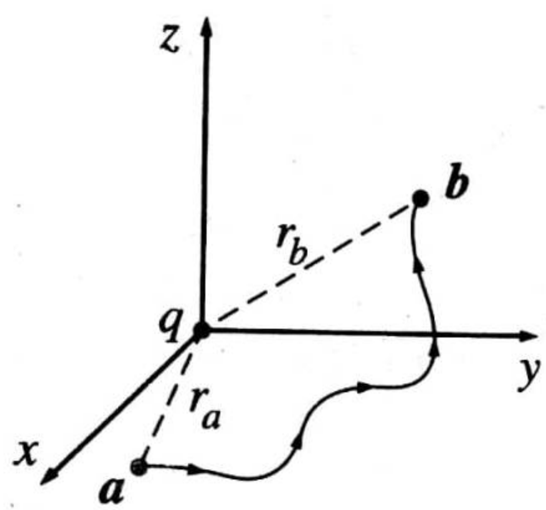


(a) (b)

At a glance it is clear from the figure that curl of this field vector has to be zero. But we ought to come up something a little more rigorous than that.

Let us first calculate the line integral of this field vector from some point $\vec{a}$ to some other point $\vec{b}$.

$$\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$$



We know, in spherical polar co-ordinate system

$$d\vec{l} = dr\,\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$$

$$\vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr$$

$$\therefore \quad \int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_{\vec{a}}^{\vec{b}} \frac{q}{r^2}dr$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

Where r_a and r_b are the distance from the origin to the point $\vec{a}$ and $\vec{b}$ respectively.

Integral around a closed path $\oint \vec{E}\cdot d\vec{l} = 0 \quad [\because \quad r_a = r_b]$

Using Stokes' theorem $\nabla \times \vec{E} = \vec{0}$

➢ Here we have assumed that the charge is situated at the origin of the co-ordinate system. But the result shows that it is independent of choice of co-ordinate system. So this equation holds good for any arbitrary choice of co-ordinates and also holds no matter where the charge is located.

➢ This equation also holds for many charges.

- ➢ According to the principle of superposition the total field at a point is a vector sum of their individual fields.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \dots$$

$$\nabla \times \vec{E} = \nabla \times \left(\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots \right)$$

$$= \nabla \times \vec{E}_1 + \nabla \times \vec{E}_2 + \nabla \times \vec{E}_3 + \ldots$$

$$= \vec{0}$$

Electric Potential

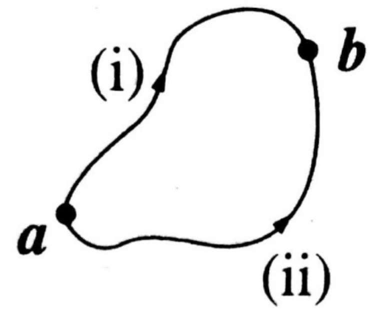
- Electric field vector $\vec{E}$ is a very special kind of vector function whose curl is always zero.

    - Example: $\vec{E} = y\hat{i}$ could not be an electrostatic field. No set of charges could produce such a field.

➢ Explore this special property of electric field function to reduce a vector problem (Finding $\vec{E}$) down to a much simpler scalar problem (Finding V).

Since $\nabla \times \vec{E} = \vec{0}$

Then $\oint \vec{E} \cdot d\vec{l} = 0$



This indicates that the line integral of $\vec{E}$ from point $\vec{a}$ to point $\vec{b}$ is the same for all paths.

✓ Otherwise we could go along path (i) and return along path (ii) and obtain $\oint \vec{E} \cdot d\vec{l} \neq 0$

Since the above integral is independent of path then we can always define scalar a function like

$$V(\vec{r}) \doteq -\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

Here O is some standard reference point on which the value of V is assumed to be zero. So V depends only on the point $\vec{r}$.

- **This $V(\vec{r})$ is called electric potential.**

Potential difference between two points $\vec{a}$ and $\vec{b}$

$$V\left(\vec{b}\right)-V\left(\vec{a}\right)=-\int_{O}^{\vec{b}}\vec{E}\cdot d\vec{l}+\int_{O}^{\vec{a}}\vec{E}\cdot d\vec{l}$$

$$=-\int_{O}^{\vec{b}}\vec{E}\cdot d\vec{l}-\int_{\vec{a}}^{O}\vec{E}\cdot d\vec{l}$$

$$=-\int_{\vec{a}}^{\vec{b}}\vec{E}\cdot d\vec{l}$$

Again from the fundamental theorem of gradients

$$V\left(\vec{b}\right)-V\left(\vec{a}\right)=\int_{\vec{a}}^{\vec{b}}\nabla V\cdot d\vec{l}$$

$$\therefore \quad \int_{\vec{a}}^{\vec{b}} \nabla V \cdot d\vec{l} = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$$

Since this is true for any points $\vec{a}$ and $\vec{b}$, the integrands must be equal

$$\vec{E} = -\nabla V$$

This is the differential version of equation $V(\vec{r}) \doteq -\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l}$

✓ If the line integral of $\vec{E}$ depended on the path taken then the definition of potential $V(\vec{r}) \doteq -\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l}$ would be nonsence. It would not define a function, since changing the path would alter the value of $V(\vec{r})$.

Example:

Which one of the below is impossible electrostatic field?

$$i)\quad \vec{E} = k\left[xy\hat{i} + 2yz\hat{j} + 3xz\hat{k}\right]$$

$$ii)\quad \vec{E} = k\left[y^2\hat{i} + \left(2xy + z^2\right)\hat{j} + 2yz\hat{k}\right]$$

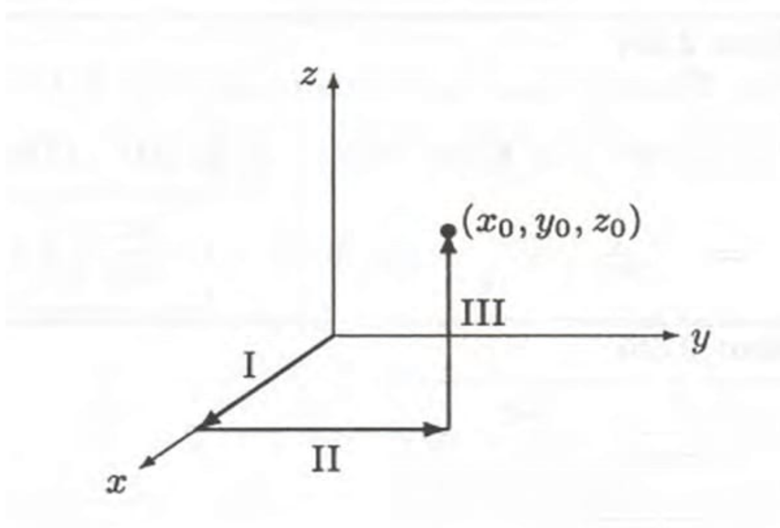
Where k is a constant.

For possible one find the potential using the origin as your reference point. Check your answer by computing $\vec{E} = -\nabla V$

Solution

$i)\quad \nabla \times \vec{E} \neq \vec{0}$ Impossible electrostatic field

$ii)\quad \nabla \times \vec{E} = \vec{0}$ Possible electrostatic field



Total path= Path I + Path II + Path III

$$\vec{E} \cdot d\vec{l} = k\left[y^2 dx + \left(2xy + z^2\right) dy + 2yz dz \right]$$

Path I $\quad y = z = 0,\ dy = dz = 0$

$$\vec{E} \cdot d\vec{l} = 0$$

Path II $\quad x = x_0,\ y: 0 \to y_0, z = 0,\ dx = dz = 0$

$$\vec{E} \cdot d\vec{l} = 2kx_0 y dy$$

$$\therefore \int_{path\,II} \vec{E}\cdot d\vec{l} = kx_0{y_0}^2$$

Path III $x = x_0,\ y = y_0, z: 0 \to z_0,\ dx = dy = 0$

$$\vec{E}\cdot d\vec{l} = 2ky_0 z\,dz$$

$$\therefore \int_{path\,III} \vec{E}\cdot d\vec{l} = ky_0{z_0}^2$$

$$Now,\ V\left(x_0, y_0, z_0\right) = -\int_0^{(x_0,y_0,z_0)} \vec{E}\cdot d\vec{l}$$

$$= \int_{path\,I} \vec{E}\cdot d\vec{l} + \int_{path\,II} \vec{E}\cdot d\vec{l} + \int_{path\,III} \vec{E}\cdot d\vec{l}$$

$$= -k\left(x_0{y_0}^2 + y_0{z_0}^2\right)$$

$$V(x,y,z) = -k\left(xy^2 + yz^2\right)$$

Check

$$-\nabla V = k\left[\frac{\partial}{\partial x}\left(xy^2+yz^2\right)\hat{i} + \frac{\partial}{\partial y}\left(xy^2+yz^2\right)\hat{j} + \frac{\partial}{\partial x}\left(xy^2+yz^2\right)\hat{k}\right]$$

$$= k\left[y^2\hat{i} + \left(2xy + z^2\right)\hat{j} + 2yz\hat{k}\right]$$

$$= \vec{E}$$

Alternatively

$$-\frac{\partial V}{\partial x} = E_x, \quad -\frac{\partial V}{\partial y} = E_y, \quad -\frac{\partial V}{\partial z} = E_z$$

$$V_1 = \int E_x dx + C_1(y,z)$$

$$V_1 = -kxy^2 + C_1(y,z)$$

$$V_2 = \int E_y dy + C_2(x,z)$$

$$V_2 = -k\left(xy^2 + yz^2\right) + C_2(x,z)$$

$$V_3 = \int E_z dz + C_3(x,y)$$

$$V_3 = -kyz^2 + C_3(x,y)$$

$$\therefore \quad V(x,y,z) = -k\left(xy^2 + yz^2\right)$$

Note

Advantages of potential formulation

Knowing V one can easily calculate $\vec{E}$ just by taking gradient

$$\vec{E} = -\nabla V$$

Note that $\vec{E}$ is a vector quantity which has three components and V is a scalar function which has one component. How can one function contain all information that three independent functions carry?

Answer: Three components of $\vec{E}$ are not really as independent as they look. They are interrelated by the condition $\nabla \times \vec{E} = \vec{0}$

In components terms

$$\frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \qquad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z} \qquad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

- Potential formulation reduces a vector problem down to a scalar problem.

Reference point

Since the choice of reference point was arbitrary the definition of potential has an ambiguity.

Changing reference points amounts to adding a constant to the potential

$$V'(\vec{r}) = -\int_{O'}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

$$= -\int_{O'}^{O} \vec{E} \cdot d\vec{l} - \int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

$$= K + V(\vec{r})$$

Adding this constant to the potential does not affect the potential difference between two points:

$$V'(\vec{b}) - V'(\vec{a}) = V(\vec{b}) - V(\vec{a})$$

$$\nabla V' = \nabla V$$

$$\vec{E}' = \vec{E}$$

Potential as such carries no real physical significance, for at any given point one can adjust its value at will by a suitable relocation of O .

Usually we choose reference point at infinity where we assume that potential is zero. $V(\propto) = 0$

<u>What happens if the charge itself extends to infinity?</u>

Example: Uniformly charged plane

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

Choose $O = \propto$

$$V(z) = -\int_{\infty}^{z} \frac{\sigma}{2\varepsilon_0} dz$$

$$= -\frac{\sigma}{2\varepsilon_0}(z - \propto)$$

Remedy?

Choose some other reference point. In this case you might use the origin.

Potential obeys the superposition principle

Total force on Q

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots$$

Dividing by Q , we get the Electric field too, obeys the superposition principle

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots$$

Integrating from the common reference point to $\vec{r}$, it follows that the potential also satisfies this superposition principle.

$$V = V_1 + V_2 + V_3 + \ldots$$

So the potential at any given point is the sum of the potentials due to all the source charges separately. This time it is an ordinary sum not a vector sum, which makes it a lot easier to work with.

Example

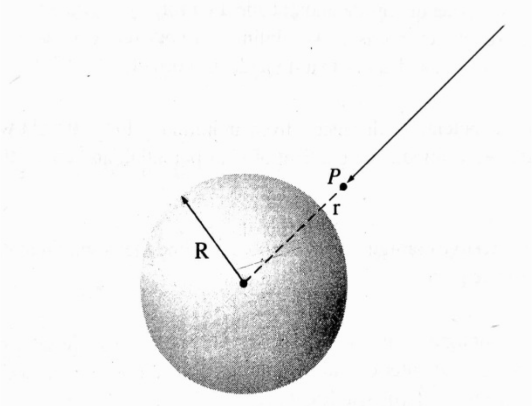
Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge.

Electric of a hollow spherical shell is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \qquad \text{for } r \geq R$$

$$= \vec{0} \qquad \text{for } r < R$$

For points outside the sphere



$$V(\vec{r}) = -\int_{O}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

$$= -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{\vec{r}} \frac{dr}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

For points inside the sphere

$$V(\vec{r}) = -\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

$$= -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{R}\frac{dr}{r^2} - \int_{R}^{r}(0)\,dr$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

Note:

Potential is not zero inside the sphere even though the field is zero. Potential is constant in this region. In calculating potential inside the sphere, you have to work in your way from infinity to that point. Do not calculate the potential inside the sphere on the basis of electric field inside only. Potential inside the sphere is very much sensitive to what is going on outside the sphere.

For example, if you put a second uniformly charged spherical shell outside the first one of radius $R' > R$,the potential inside R would change, even though the field would still be zero.

From Gauss's law we know that charge exterior to a given point produces no net field at that point, provided it is spherically or cylindrically symmetric. But there is no such rule for potential, when infinity is used as the reference point.

Example

Find the potential inside and outside a uniformly charged solid sphere whose radius is R and total charge is q . Use infinity as your reference.

Electric of a uniformly charged solid sphere is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \qquad \text{for } r > R$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{R^3} r\hat{r} \qquad \text{for } r < R$$

For points outside the sphere $r > R$

$$V(\vec{r}) = -\int_O^{\vec{r}} \vec{E}\cdot d\vec{l} = -\frac{q}{4\pi\varepsilon_0}\int_\infty^{\vec{r}} \frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

For points inside the sphere $r < R$

$$V(\vec{r}) = -\int_{O}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

$$= -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{R}\frac{dr}{r^2} - \frac{q}{4\pi\varepsilon_0}\int_{R}^{r}\frac{1}{R^3}rdr$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2 - R^2}{2}\right)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

Example

Find the potential a distance s from an infinitely long straight wire that carries a uniform line charge λ .

Electric field due an infinitely long straight wire is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{s}\hat{s}$$

Reference point ???

$$s = \infty$$

We can not choose $s = \infty$ as reference point as the charge itself extends to infinity.

Let us choose $s = a$ as the reference point.

$$\therefore \quad V(s) = -\int_a^s \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{s}ds$$

$$= \frac{1}{4\pi\varepsilon_0} 2\lambda \ln\left(\frac{a}{s}\right)$$

- It is clear that why we did not choose $s = \infty$ as reference point.
- Similarly one can not choose the reference point at $s = 0$.

Poisson’s equation and Laplace’s equation

We know that $\vec{E} = -\nabla V$

The fundamental equations for $\vec{E}$ are

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \& \qquad \nabla \times \vec{E} = \vec{0}$$

These equations in terms of V

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ This is known as Poisson’s equation

In region where there is no charge $(\rho = 0)$

$$\nabla^2 V = 0$$ This is known as Laplace’s equation

We require only one differential equation to determine V.

Potential of a localized charge distribution

- We have defined V in terms of $\vec{E}$ as $V(\vec{r}) = -\int_{O}^{\vec{r}} \vec{E} \cdot d\vec{l}$
- But our main aim in electrostatics is to calculate $\vec{E}$
- If we already know $\vec{E}$ then why do we invest time to calculate V ?
- The idea of potential formulation is that it is easy to get V and then $\vec{E}$ can be calculated by taking gradient of V , $\vec{E} = -\nabla \cdot V$
- So V is to be calculated by knowing charge density ρ
- Poisson's equation relates between V and ρ . But it gives ρ if we know V
- We have to invert the Poisson's equation. We shall do it in this section.

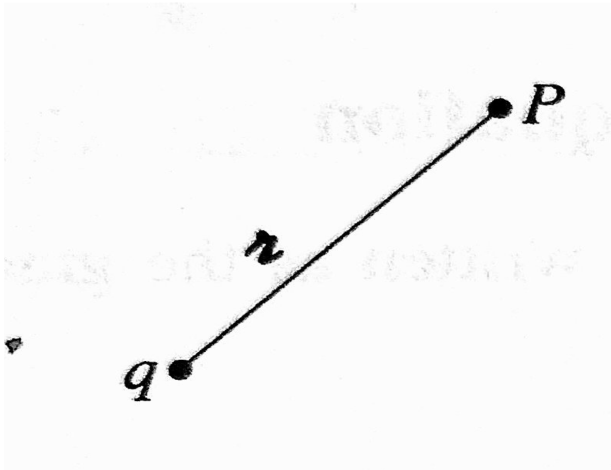
The potential at a point due to a single point charge q situated at the origin

$$V(\vec{r}) = -\int_O^{\vec{r}} \vec{E} \cdot d\vec{l} = -\frac{q}{4\pi\varepsilon_0}\int_\infty^{\vec{r}} \frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

✓Reference point is at infinity

In general potential of a point charge q

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\Re}$$

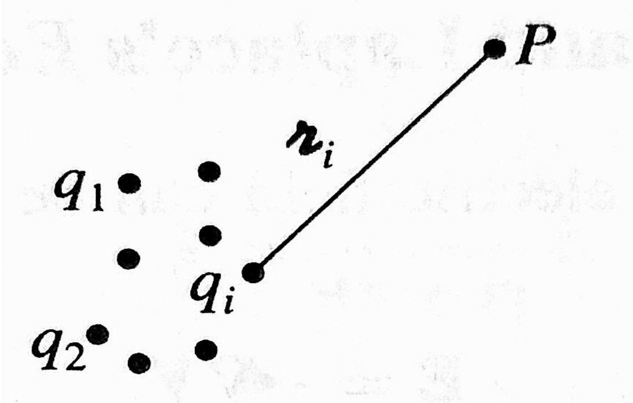


Where $\Re$ is the distance from the charge to $\vec{r}$

Potential of a collection of point charges

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\Re_i}$$

✓ We have used superposition principle

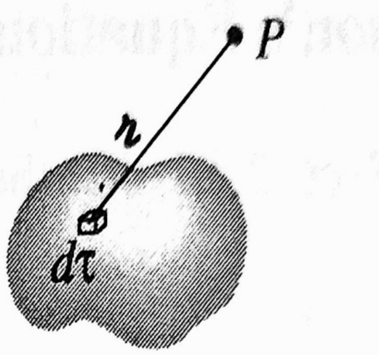


Potential of a continuous distribution of charges

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{\Re}dq$$

For a volume charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}\,')}{\Re}d\tau\,'$$



➢Using this relation, potential of a charge distribution can be calculated knowing the charge density ρ

➢This is the solution to the Poisson's equation for a localized charge distribution.

➢ Compare the following two equations

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\Re} d\tau' \qquad \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{\Re^2}\hat{\Re} d\tau'$$

In the first expression for calculating $V(\vec{r})$ the unit vector $\hat{\Re}$ is missing, so we do not need to think about components . As a result calculation of $V(\vec{r})$ is more easier than calculating $\vec{E}(\vec{r})$

Potential for line charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(\vec{r}\,')}{\Re} dl'$$

Potential for surface charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(\vec{r}\,')}{\Re} da'$$

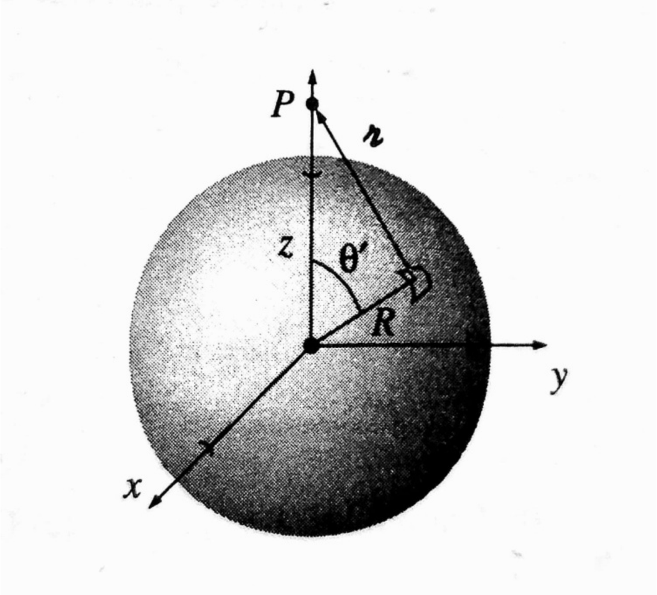
For a volume charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{\Re} d\tau'$$

- All the above three cases we assume infinity as the reference point. So if the charge itself extends to infinity then the integral will diverge.

Example

Find the potential of a uniformly charged spherical shell of radius R



Solution

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{\mathfrak{R}} ds'$$

Set the point $\vec{r}$ on the z axis.

$$\mathfrak{R}^2 = R^2 + z^2 - 2RZ\cos\theta'$$

So $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{\sqrt{R^2 + z^2 - 2RZ\cos\theta'}} R^2 \sin\theta' d\theta' d\phi'$$

$$V\left(\vec{r}\right)=\frac{1}{4\pi\varepsilon_0}\int\frac{\sigma}{\sqrt{R^2+z^2-2RZ\cos\theta'}}R^2\sin\theta'd\theta'd\phi'$$

$$=\frac{1}{4\pi\varepsilon_0}2\pi R^2\sigma\int_0^{\pi}\frac{\sin\theta'd\theta'}{\sqrt{R^2+z^2-2RZ\cos\theta'}}$$

$$=\frac{1}{4\pi\varepsilon_0}2\pi R^2\sigma\left(\frac{1}{Rz}\sqrt{R^2+z^2-2RZ\cos\theta'}\right)\Bigg|_0^{\pi}$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{2\pi R\sigma}{z}\left(\sqrt{R^2+z^2+2RZ}-\sqrt{R^2+z^2-2RZ}\right)$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{2\pi R\sigma}{z}\left(\sqrt{\left(R+z\right)^2}-\sqrt{\left(R-z\right)^2}\right)$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{2\pi R\sigma}{z}\left(\left(R+z\right)-\left|R-z\right|\right)$$

- Outside the sphere

$$V(z)=\frac{R\sigma}{2\varepsilon_0 z}\left[(R+z)-(z-R)\right]$$

$$V(z)=\frac{R^2\sigma}{\varepsilon_0 z}$$

- Inside the sphere

$$V(z)=\frac{R\sigma}{2\varepsilon_0 z}\left[(R+z)-(R-z)\right]$$

$$V(z)=\frac{R\sigma}{\varepsilon_0}$$

In terms of q

$$V(\vec{r})=\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\,,\quad \text{for points outside the sphere}$$

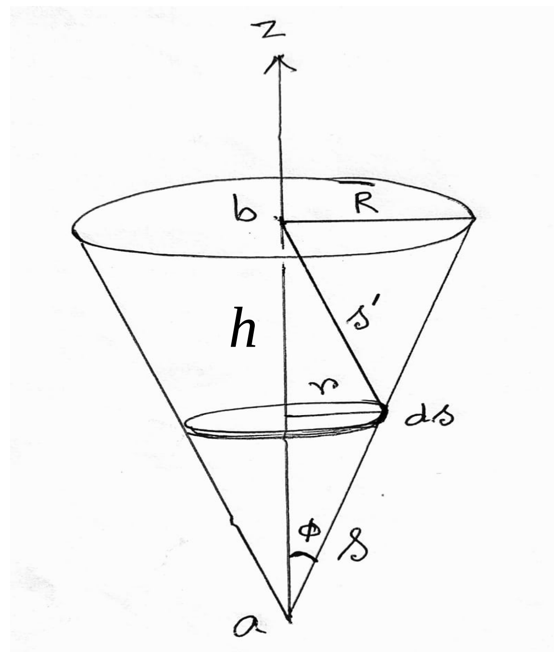
$$=\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\,,\quad \text{for points inside the sphere}$$

Problem

A conical surface carries a uniform surface charge σ . The height of the cone is h and the radius of the top is R . Find the potential difference between points $\vec{a}$ (the vertex) and $\vec{b}$ (the centre of the top).

Solution

From the figure

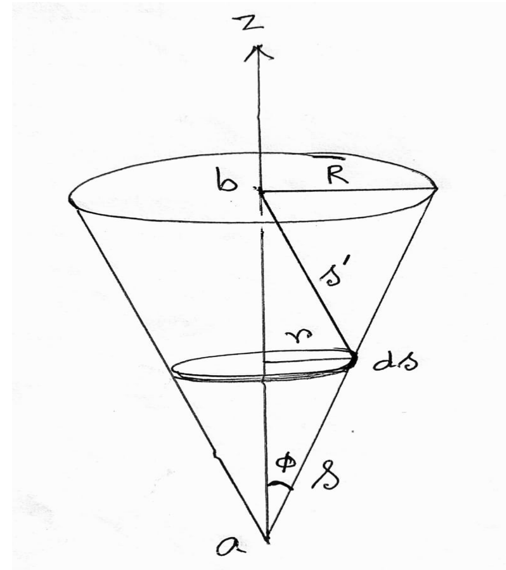


$$\frac{R}{r} = \frac{\sqrt{h^2 + R^2}}{s} \qquad \Rightarrow r = \frac{sR}{\sqrt{h^2 + R^2}}$$

$$V(a) = \frac{1}{4\pi\varepsilon_0} \int_0^{\sqrt{h^2+R^2}} \frac{(2\pi r ds)\sigma}{s}$$

$$V(a)=\frac{1}{4\pi\varepsilon_0}2\pi\sigma\frac{R}{\sqrt{h^2+R^2}}\int_0^{\sqrt{h^2+R^2}} ds$$

$$V(a)=\frac{\sigma R}{2\varepsilon_0} \quad \ldots\ldots\ldots\ldots\ldots (1)$$



Again from the figure $\cos\phi=\frac{h}{\sqrt{h^2+R^2}}=t(\text{say})$

$$s'^2=h^2+s^2-2hs\cos\phi$$

$$s'^2=h^2+s^2-2hts$$

$$V(b)=\frac{1}{4\pi\varepsilon_0}\int_0^{\sqrt{h^2+R^2}}\frac{(2\pi r ds)\sigma}{s'}$$

$$V(b) = \frac{1}{4\pi\varepsilon_0} \int_0^{\sqrt{h^2+R^2}} \frac{(2\pi r ds)\sigma}{s'}$$

$$V(b) = \frac{1}{4\pi\varepsilon_0} 2\pi\sigma \frac{R}{\sqrt{h^2+R^2}} \int_0^{\sqrt{h^2+R^2}} \frac{sds}{\sqrt{h^2+s^2-2hts}}$$

$$= \frac{\sigma R}{2\varepsilon_0} \frac{1}{\sqrt{h^2+R^2}} \int_0^{\sqrt{h^2+R^2}} \frac{sds}{\sqrt{h^2+s^2-2hts}}$$

$$= \frac{\sigma R}{2\varepsilon_0} \frac{1}{\sqrt{h^2+R^2}} \int_0^{\sqrt{h^2+R^2}} \frac{sds}{\sqrt{h^2+s^2-2hts}} \qquad \ldots\ldots\ldots\ldots(2)$$

Consider the integration

$$I = \int \frac{sds}{\sqrt{h^2+s^2-2hts}}$$

$$I = \int \frac{(s-ht)+ht}{\sqrt{h^2+s^2-2hts}}ds$$

$$\text{let } z^2 = h^2+s^2-2hts$$

$$zdz = (s-ht)ds$$

$$I = \int \frac{zdz}{z} + ht\int \frac{ds}{\sqrt{h^2+s^2-2hts}}$$

$$= z + ht\int \frac{ds}{\sqrt{(s-ht)^2+(h^2-h^2t^2)}}$$

$$= \sqrt{h^2+s^2-2hts} + ht\ln\left[(s-ht)+\sqrt{h^2+s^2-2hts}\right]$$

From equation 2

$$V(b)=\frac{\sigma R}{2\varepsilon_0}\frac{1}{\sqrt{h^2+R^2}}\int_0^{\sqrt{h^2+R^2}}\frac{sds}{\sqrt{h^2+s^2-2hts}}\qquad ....(2)$$

$$=\frac{\sigma R}{2\varepsilon_0}\frac{1}{\sqrt{h^2+R^2}}\left[\sqrt{h^2+s^2-2hts}+ht\ln\left\{(s-ht)+\sqrt{h^2+s^2-2hts}\right\}\right]_0^{\sqrt{h^2+R^2}}$$

$$=\frac{\sigma R}{2\varepsilon_0}\frac{1}{\sqrt{h^2+R^2}}\left[R+ht\ln\left(\sqrt{h^2+R^2}-ht+R\right)-h-ht\ln(h-ht)\right]$$

$$=\frac{\sigma R}{2\varepsilon_0}\left[\frac{R-h}{\sqrt{h^2+R^2}}+\frac{h^2}{h^2+R^2}\ln\left(\frac{\sqrt{h^2+R^2}-\frac{h^2}{\sqrt{h^2+R^2}}+R}{h-\frac{h^2}{\sqrt{h^2+R^2}}}\right)\right]$$

$$V(b)=\frac{\sigma R}{2\varepsilon_0}\left[\frac{R-h}{\sqrt{h^2+R^2}}+\frac{h^2}{h^2+R^2}\ln\left(\frac{R^2+R\sqrt{h^2+R^2}}{h\sqrt{h^2+R^2}-h^2}\right)\right]$$

$$\therefore\ V(a)-V(b)$$

$$=\frac{\sigma R}{2\varepsilon_0}\left[1-\frac{R-h}{\sqrt{h^2+R^2}}-\frac{h^2}{h^2+R^2}\ln\left(\frac{R^2+R\sqrt{h^2+R^2}}{h\sqrt{h^2+R^2}-h^2}\right)\right]$$

Special case if $h=R$

$$\therefore\ V(a)-V(b)$$

$$=\frac{\sigma R}{2\varepsilon_0}\left[1-\frac{1}{2}\ln\left(\frac{\sqrt{2}R^2+R^2}{\sqrt{2}R^2-R^2}\right)\right]$$

$$V(a)-V(b)=\frac{\sigma R}{2\varepsilon_0}\left[1-\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right]$$

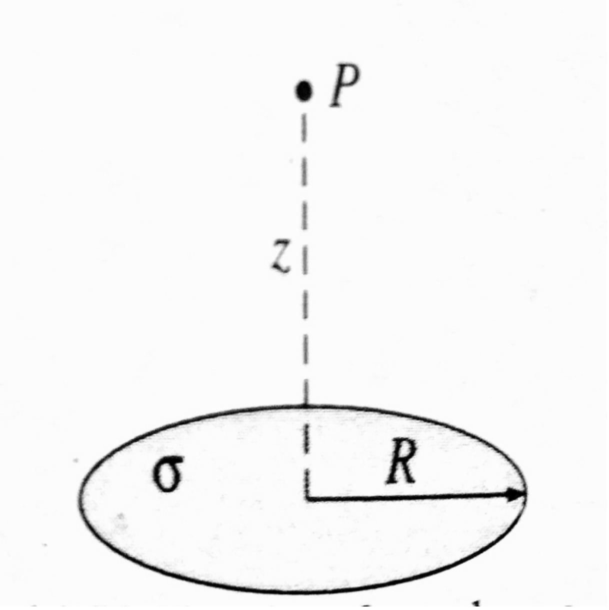
$$=\frac{\sigma R}{2\varepsilon_0}\left[1-\frac{1}{2}\ln\left(\frac{\left(\sqrt{2}+1\right)^2}{1}\right)\right]$$

$$\therefore \quad V(a)-V(b)=\frac{\sigma R}{2\varepsilon_0}\left[1-\ln\left(1+\sqrt{2}\right)\right]$$

Problem

Find the potential at a distance z above the centre of a circular disk of radius R surface charge density σ . Compute $\vec{E} = -\nabla V$

Solution



$$V(z) = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{(2\pi r dr)\sigma}{\sqrt{r^2+z^2}}$$

$$= \frac{1}{4\pi\varepsilon_0} 2\pi\sigma\left(\sqrt{r^2+z^2}\right)_0^R$$

$$\therefore \quad V(z) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2+z^2} - z\right)$$

➢ Electric field

$$\vec{E} = -\frac{\partial V}{\partial z}\hat{z} \qquad \left[\because \quad \frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0\right]$$

$$\because \; V(z) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{R^2 + z^2} - z\right)$$

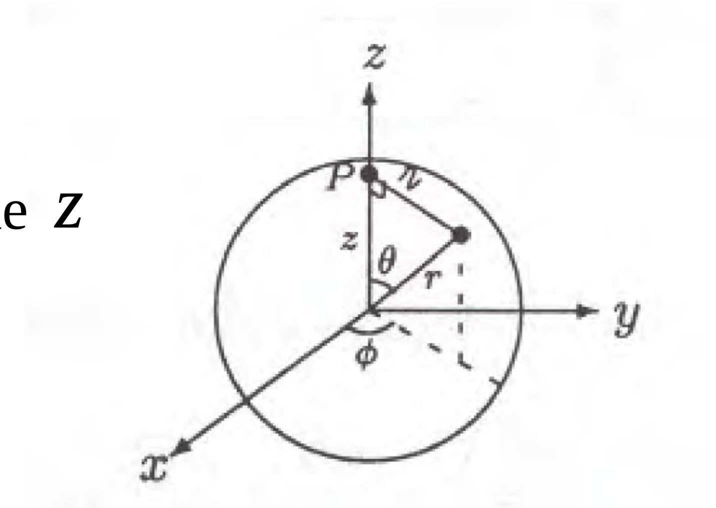
$$\vec{E} = -\frac{\sigma}{2\varepsilon_0}\left\{\frac{1}{2}\frac{1}{\sqrt{R^2 + z^2}}2z - 1\right\}\hat{z}$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\left\{1 - \frac{z}{\sqrt{R^2 + z^2}}\right\}\hat{z}$$

Problem

Calculate the potential inside a uniformly charged solid sphere of radius R and total charge q.

Solution



Choose axes such that the point P lies on the z axis.

$$V(z) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{\Re} d\tau$$

$$\Re = \sqrt{z^2 + r^2 - 2rz\cos\theta}$$

$$V(z) = \frac{1}{4\pi\varepsilon_0}\int \frac{r^2 \sin\theta d\theta d\phi dr}{\sqrt{z^2 + r^2 - 2rz\cos\theta}}$$

Now $\int_0^{2\pi} d\phi = 2\pi$

$$\int_0^{\pi} \frac{\sin\theta d\theta}{\sqrt{z^2+r^2-2rz\cos\theta}} = \frac{1}{rz}\left[\sqrt{z^2+r^2-2rz\cos\theta}\right]_0^{\pi}$$

$$= \frac{1}{rz}\left(\sqrt{z^2+r^2+2rz}-\sqrt{z^2+r^2-2rz}\right)$$

$$= \frac{1}{rz}\left(r+z-\left|r-z\right|\right)$$

$$= \begin{cases} \dfrac{2}{z}, \text{ if } r < z \\ \dfrac{2}{r}, \text{ if } r > z \end{cases}$$

$$V(z) = \frac{1}{4\pi\varepsilon_0} \int \frac{r^2 \sin\theta d\theta d\phi dr}{\sqrt{z^2 + r^2 - 2rz\cos\theta}}$$

$$V(z) = \frac{1}{4\pi\varepsilon_0} 2\pi \cdot 2 \left[\int_0^z \frac{r^2}{z} dr + \int_z^R \frac{r^2}{r} dr \right]$$

$$= \frac{\rho}{2\varepsilon_0} \left(R^2 - \frac{z^2}{3} \right)$$

Now $\rho = \dfrac{q}{\frac{4}{3}\pi R^3}$

$$V(z) = \frac{q}{8\pi\varepsilon_0 R} \left(3 - \frac{z^2}{R^2} \right)$$

$$V(\vec{r}) = \frac{q}{8\pi\varepsilon_0 R} \left(3 - \frac{r^2}{R^2} \right)$$

Reference:

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