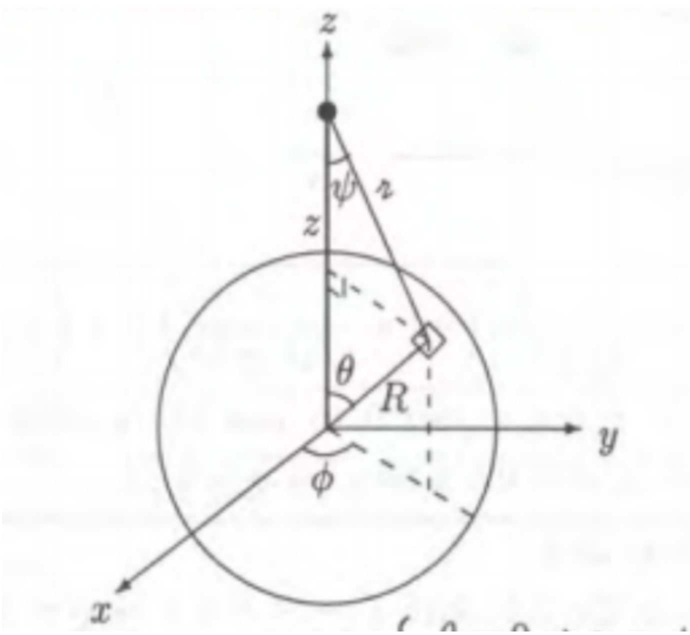
Gauss's Law in Electrostatics: Introduction

Shyamal Bhar

Assistant Professor
Department of Physics
Vidyasagar College for Women
Kolkata - 6

Consider few problems

Problem # 1:
Find the electric field a distance Z from the center of a spherical surface of radius R , which carries a uniform charge density σ .



- **Direction of Electric field**: **Along z axis**
- **Elemental charge on the surface of area da-**

$$dq = \sigma da = \sigma R^2 \sin\theta d\theta d\varphi$$

- **From the diagram:**

$$\mathcal{R}^2 = R^2 + z^2 - 2zR\cos(\theta)$$

$$\cos\psi = \frac{z - R\cos\theta}{\mathcal{R}}$$

$$E_z = \frac{1}{4\pi\varepsilon_0}\int_0^{\pi}\frac{\sigma R^2 \sin\theta d\theta\left(z - R\cos\theta\right)}{\left(R^2 + z^2 - 2zR\cos\theta\right)^{3/2}}\int_0^{2\pi} d\phi$$

$$= \frac{1}{4\pi\varepsilon_0}\left(2\pi R^2\sigma\right)\int_0^{\pi}\frac{\left(z - R\cos\theta\right)\sin\theta d\theta}{\left(R^2 + z^2 - 2zR\cos\left(\theta\right)\right)^{3/2}}$$

Let $R^2 + z^2 - 2zR\cos\theta = p^2$

$$2zR\sin\theta d\theta = 2pdp$$

$$\sin\theta d\theta = \frac{pdp}{zR}$$

Now $$2z^2 - 2zR\cos\theta = p^2 + z^2 - R^2$$

$$\left(z - R\cos\theta\right) = \frac{p^2 + z^2 - R^2}{2z}$$

$$E = \frac{1}{4\pi\varepsilon_0}\left(2\pi R^2\sigma\right)\int_{|z-R|}^{z+R} \frac{p\,dp}{zR}\,\frac{(p^2 + z^2 - R^2)}{2zp^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\left(2\pi R^2\sigma\right)}{2z^2R}\int_{|z-R|}^{z+R}\left(1 + \frac{z^2 - R^2}{p^2}\right)dp$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\left(2\pi R^2\sigma\right)}{2z^2R}\left[p - \frac{z^2 - R^2}{p}\right]_{|z-R|}^{z+R}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{\left(2\pi R^2 \sigma\right)}{2z^2 R} \left[(z+R) - \frac{\left(z+R\right)\left(z-R\right)}{\left(z+R\right)} - \left|z-R\right| + \frac{\left(z+R\right)\left(z-R\right)}{\left|z-R\right|} \right]$$

- **Outside the sphere** $z > R$ **:**

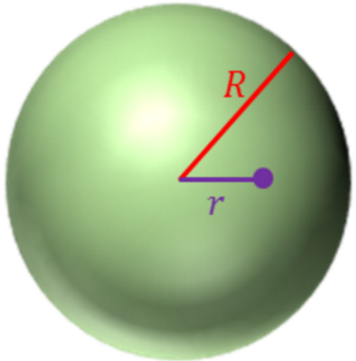
$$E_z = \frac{1}{4\pi\varepsilon_0} \frac{\left(2\pi R^2 \sigma\right)}{2z^2 R} 4R$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{z^2} \hat{z}$$

- **Inside the sphere** $z < R$ **:**

$$\vec{E} = \vec{0}$$

Problem # 2:
Find the electric field inside and outside a sphere of radius R , which carries a uniform volume charge density ρ .



- **Using previous results:**

- **All the points (shells) interior to the given point i.e. smaller than r contribute to the electric filed as all the charge were concentrated at the centre of the sphere.**

- **While all the exterior shell contribute nothing to the electric field.**

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q_{\text{int}}}{r^2} \hat{r}$$

q_{int} **is the total charge interior to the point (interior shells)**

- **Outside the sphere :**

  - **All charge is interior**

$$q_{int} = q$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

- **Inside the sphere :**

  - **Only a fraction of the total charge is interior to the points considered**

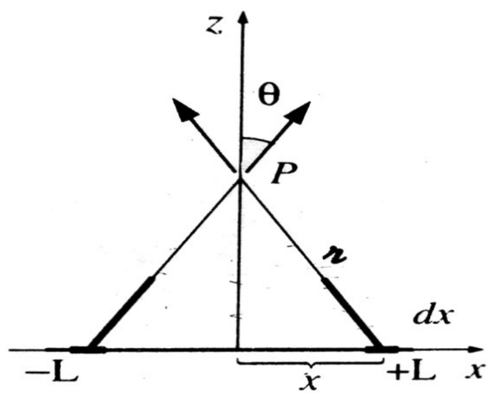
$$q_{int} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} q = \frac{r^3}{R^3} q$$

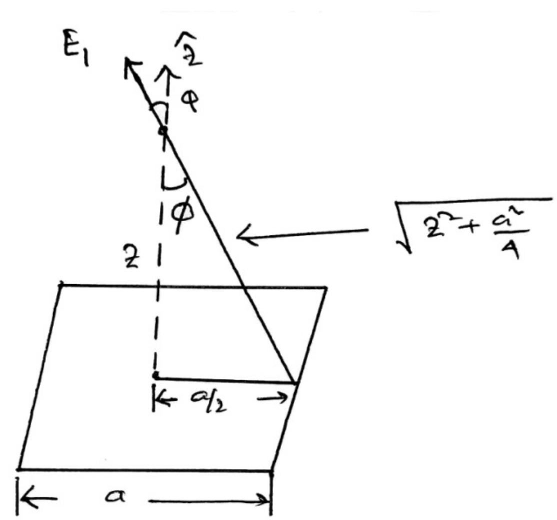
$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{\frac{r^3}{R^3} q}{r^2} \hat{r}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^3} \hat{r}$$

Problem # 3: Find the electric field at a distance Z above the centre of a square loop of side a carrying uniform line charge λ

- **Using results:**





$$E_z = \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}},$$

Here $2L = a$

$$L = \frac{a}{2}$$

$$z \to \sqrt{z^2 + \frac{a^2}{4}}$$

$$E_z = E_1 \times 4\cos\phi$$

$$\cos\phi = \frac{z}{\sqrt{z^2 + \frac{a^2}{4}}}$$

$$E_z = \frac{1}{4\pi\varepsilon_0} \frac{2\lambda \frac{a}{2}}{\sqrt{z^2 + \frac{a^2}{4}}\sqrt{z^2 + \frac{a^2}{4} + \frac{a^2}{4}}} \times 4 \times \frac{z}{\sqrt{z^2 + \frac{a^2}{4}}}$$

$$E_z = \frac{1}{4\pi\varepsilon_0} \frac{4\lambda a z}{\left(z^2 + \frac{a^2}{4}\right)\sqrt{z^2 + \frac{a^2}{2}}}$$

Gauss's Law: Divergence and Curl of Electrostatics fields

- Our main job in Electrostatics is to calculate the electric field due a point charge or charge distribution.

- We have seen that the integral involving in calculating electric field due to a spherical charge distribution is formidable even for reasonably simple charge distribution.

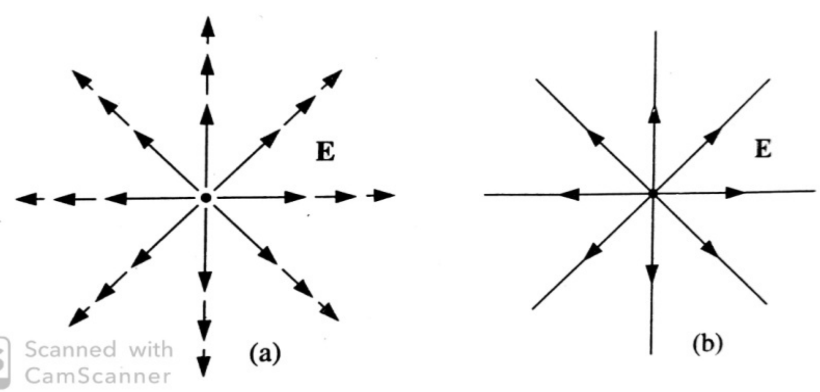
- Here we shall learn some tools and tricks to avoid these integrals. These are the divergence and Curl of $\vec{E}$.

- ❑ **Before we going to calculate divergence of directly, we shall discuss it qualitatively first.**

- ❑ **Consider a simplest charge distribution, a single point charge located at the origin.**

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}\hat{r}$$

Representation of this field graphically



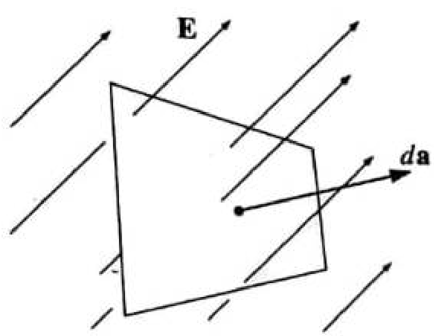
Density of field lines

$$D = \frac{n}{4\pi r^2}$$

Flux of the electric field:

$$\phi_E = \int \vec{E} \cdot d\vec{S}$$

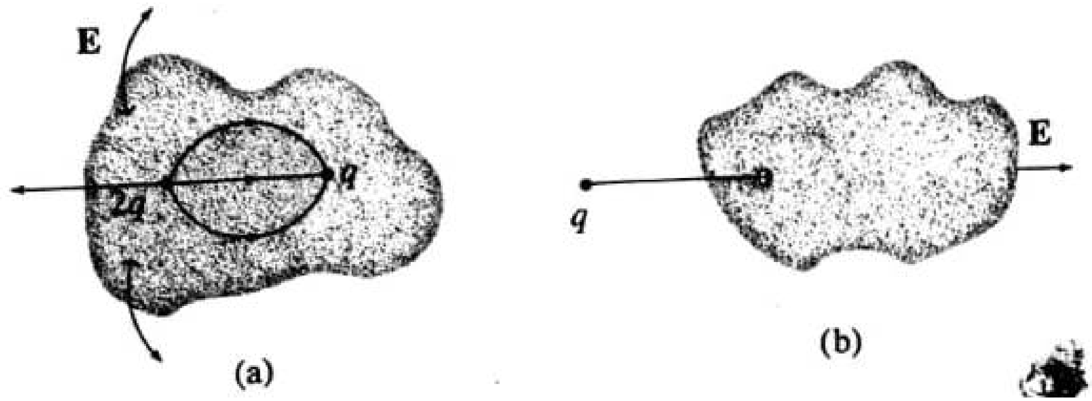
The flux of $\vec{E}$ through a surface S which is the number of field lines passing through S



- For a given sampling rate the flux is proportional to the number of field lines drawn because the field strength is proportional to the density of the field line and hence $\vec{E} \cdot d\vec{S}$ is proportional to the number of lines passing through the infinitesimal surface area

- This indicates that the flux through any closed surface is a measure of the total charge inside.

$$\phi_E \propto q_{enc}$$



➢ **Field lines that originate on a positive charge must either pass out through the surface or terminate on a negative charge inside.**

➢**And a charge outside the surface will contribute nothing to the flux, since its field lines pass in one side and out at other.**

$$\vec{E} = \vec{0}$$

If charge is outside the surface

➢ **This is the essence of Gauss's law**.

Quantitative calculation:

Example -1: Consider a point charge q at the origin (for a Spherical surface).

The flux of $\vec{E}$ through a sphere of radius r is

$$\phi = \oint \vec{E} \cdot d\vec{s}$$

$$= \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot\left(r^2 \sin\theta d\theta d\phi \hat{r}\right)$$

$$= \frac{q}{\varepsilon_0}$$

✓ **The radius of the sphere cancels out. The field falls off as $\frac{1}{r^2}$ while the surface area goes up r^2 and so the product is constant.**

✓**In field line picture, the number of filed lines passing through any sphere centered at the origin, regardless of its size remain same. Hence the flux is constant.**

✓ **Any arbitrary surface, whatever its shape, would contain same number of field lines. So the flux through any sphere enclosing the charge is $\frac{q}{\varepsilon_0}$**

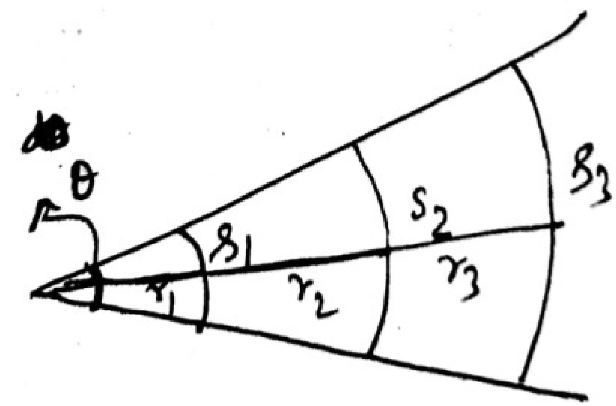
Example -2: Consider a point charge q at the origin (for arbitrary shaped surface)

Pre-requisite: Solid Angle

Angle- Given two intersecting lines or line segments, the amount of rotation about the point of intersection (the vertex) required to bring one into correspondence with the other is called the angle between them.

Sometimes it is called plane angle to distinguish it from solid angle.

$$\frac{s_1}{r_1} = \frac{s_2}{r_2} = \frac{s_3}{r_3} = \text{Const.} = \theta$$

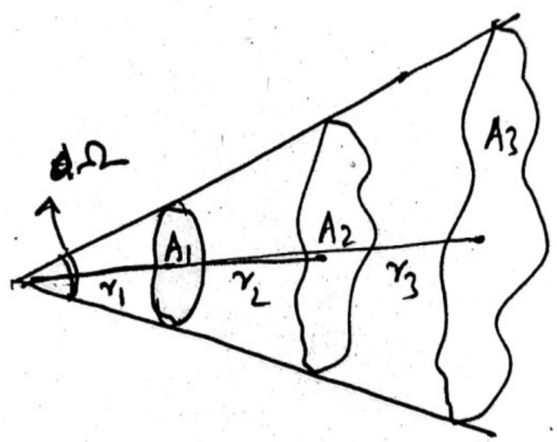


This constant quantity is called angle subtended by the arc at the centre of the circle (Apex).

So angle is the ratio of subtended arc length on circle to the radius.

A circle has total 2π radian angle.

Solid Angle



$$\frac{A_1}{r_1^2} = \frac{A_2}{r_2^2} = \frac{A_3}{r_3^2} = \text{Const.} = \Omega$$

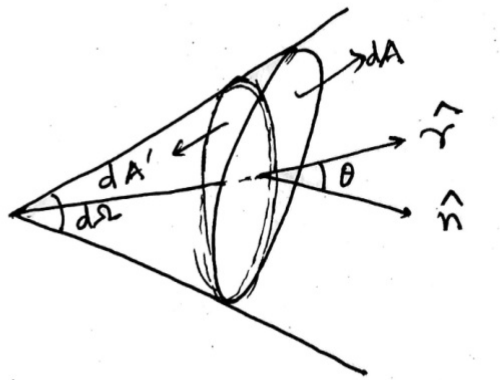
Here the direction of area vector ($\hat{n}$) and radius vector $\hat{r}$ are same.

This constant quantity is called an solid angle subtended by the area A at the centre

$$d\Omega = \frac{dA}{r^2}$$ when $\hat{n}$ and $\hat{r}$ are in same direction

Solid angle is the ratio of subtended area on sphere to radius squared

When $\hat{n}$ and $\hat{r}$ are in same direction



$$d\Omega = \frac{dA'}{r^2}$$

$$dA' = dA\cos\theta$$

$$\therefore \ d\Omega = \frac{dA\cos\theta}{r^2}$$

$$= \frac{dA\,\hat{n}\cdot\hat{r}}{r^2}$$

$$= \frac{d\vec{A}\cdot\hat{r}}{r^2}$$

The solid angle Ω subtended by a surface A is defined as the surface area of a unit sphere covered by the surface's projection onto the sphere.

$$\Omega = \iint \frac{d\vec{A} \cdot \hat{r}}{r^2}$$

$$\Omega = \oiint d\Omega = \oiint \frac{d\vec{A} \cdot \hat{r}}{r^2} = 4\pi$$

For a closed surface

The electric field at a point due to a single point charge located at the origin

$$\vec{E} = \frac{1}{4\pi\varepsilon_o} \frac{q}{r^2} \hat{r}$$

Flux of the electric field

$$\phi = \oint_S \vec{E} \cdot d\vec{s}$$

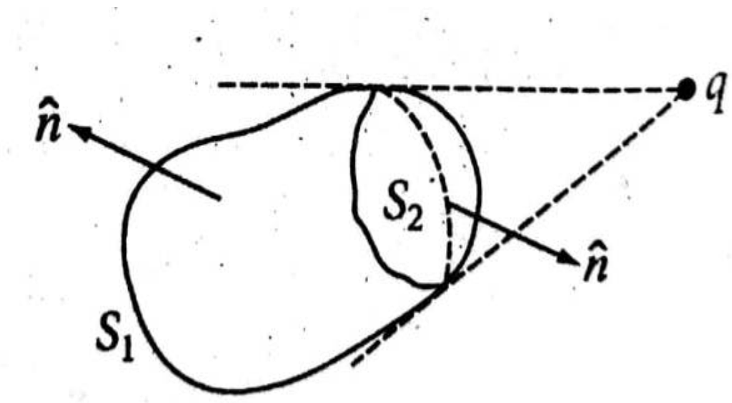
$$= \frac{q}{4\pi\varepsilon_0} \oint_S \frac{\hat{r} \cdot d\vec{s}}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0} \oint_S d\Omega$$

$$= \frac{q}{4\pi\varepsilon_0} 4\pi$$

$$= \frac{q}{\varepsilon_0}$$

Charge is outside the surface



$$\phi = \oint_S \vec{E} \cdot d\vec{s}$$

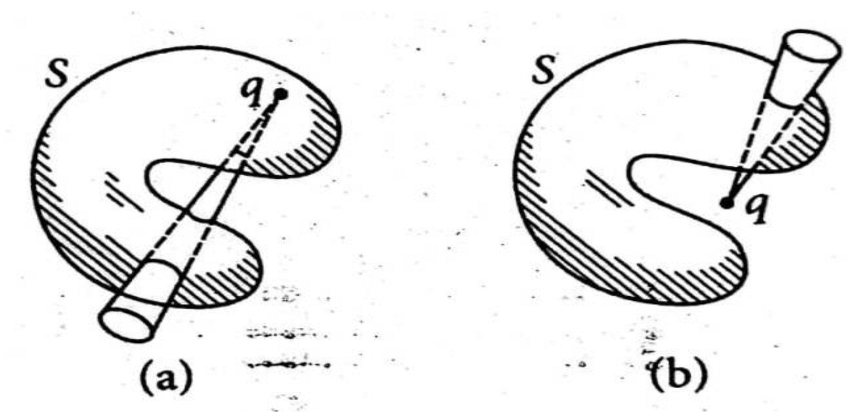
$$= \frac{q}{4\pi\varepsilon_0}\left[\oint_{S_1} \frac{\hat{r} \cdot d\vec{s}}{r^2} + \oint_{S_2} \frac{\hat{r} \cdot d\vec{s}}{r^2}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\oint_{S_1} d\Omega + \oint_{S_2} d\Omega\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[4\pi + (-4\pi)\right]$$

$$= 0$$

Re-entrant surface



✓ **Point charge is inside the re-entrant surface, a cone with its apex at the point charge intersects the surface an odd number of times**

✓**Of these only one cut contribute to the net outward flux. The remaining even number of cuts contribute nothing since for half of these cuts flux is outward and for remaining half it is inward, so the sum is zero.**

✓ **The charge is outside the re-entrant surface, a cone with its apex at the point charge intersects the surface an even number of times. Flux is zero.**

For many point charges

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i$$

Flux

$$\phi = \oint_s \vec{E} \cdot d\vec{s}$$

$$= \sum_{i=1}^{n} \left(\oint_s \vec{E}_i \cdot d\vec{s} \right)$$

$$= \sum_{i=1}^{n} \left(\frac{q_i}{\varepsilon_0} \right)$$

$$= \frac{Q_{enc}}{\varepsilon_0}$$

So for any closed surface

$$\oint_s \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon_0}$$

where Q_{enc} is the total charge enclosed within the surface S.

So, Gauss's law states that total outward flux of the electric field through a closed surface is equal to the total charge enclosed by the surface divided by the free space permittivity.

Differential equation of Gauss's law

Applying divergence theorem, we can write

$$\oint_s \vec{E} \cdot d\vec{s} = \iiint \nabla \cdot \vec{E} \, dv$$

$$\iiint \nabla \cdot \vec{E} \, dv = \frac{Q_{enc}}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0} \iiint \rho \, dv$$

$$= \iiint \left(\frac{\rho}{\varepsilon_0} \right) dv$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

This is the Gauss's law in differential form.