Numerical Solution of ODE (1^{st} Order)

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Different Numerical Methods

- Euler Method
- Modified Euler Method
- 2^{nd} Order Runge-Kutta Method (RK2)
- 4^{th} Order Runge-Kutta Method (RK4)

Euler Method

General form of 1st Order Ordinary Differential Equation takes the form

$$\frac{dy}{dx} = f(x, y) \qquad \qquad ....(1)$$

With initial condition $y(x = x_0) = y_0$ [Initial value problem]

Equation (1) can be written as (in integral form)

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x, y)\,dx \qquad ....(2)$$

x_{n+1} *and* x_n are very close to each other such that $x_{n+1} - x_n = h$

Euler Method Contd…..

Now, this integral equation (2) can be solved by using different Approximation method. In the first Approximation we can evaluate the integration of equation (2) by assuming the function $f(x, y) = f(x_n, y_n)$, then we get

$$y_{n+1} = y_n + h\,f\left(x_n, y_n\right) \qquad \text{..........}(3)$$

This is called Euler's Formula.

Modified Euler Method

Approximation Scheme:

If we approximate the integration of equation (2) by Trapezoidal method , then the equation takes the form

$$y_{n+1} = y_n + \frac{h}{2}\left[f\left(x_n, y_n\right) + f\left(x_{n+1}, y_{n+1}\right)\right]$$

Use Euler method to Approximate $y_{n+1} \approx y_n + h\ f(x_n, y_n)$ in the above equation we get

$$y_{n+1} = y_n + \frac{h}{2}\left[f\left(x_n, y_n\right) + f\left(x_{n+1},\ y_n + h\, f\left(x_n, y_n\right)\right)\right]$$

Modified Euler Method Contd……

Let
$$k_1 = h\, f\left(x_n , y_n\right)$$
$$k_2 = h\, f\left(x_{n+1} , y_n + k_1\right)$$

So, the above equation becomes

$$y_{n+1} = y_n + \frac{k_1 + k_2}{2}$$

This is the modified Euler's Formula.

Modified Euler Method Contd……

Local truncation Error due to the approximation:

$$y_{n+1} = y_n + \frac{k_1 + k_2}{2}$$

with $k_1 = h\,f\left(x_n, y_n\right) \quad and \quad k_2 = h\,f\left(x_n + h, y_n + k_1\right)$

Taylor series expansion of k_2

$$k_2 = h\left[\,f\left(x_n, y_n\right) + h\frac{\partial}{\partial x} f\left(x_n, y_n\right) + k_1\frac{\partial}{\partial y} f\left(x_n, y_n\right) + O(h^2, k_1^{\,2})\right]$$

Modified Euler Method Contd……

$$Since,\ k_1 = h\, f\left(x_n, y_n\right) = O\left(h\right)$$

$$\therefore \quad \frac{1}{2}\left(k_1 + k_2\right) = h\, f\left(x_n, y_n\right) + \frac{h^2}{2}\left[\frac{\partial}{\partial x} f\left(x_n, y_n\right) + f\left(x_n, y_n\right)\frac{\partial}{\partial y} f\left(x_n, y_n\right)\right] + O(h^3) \quad .....(4)$$

Taylor series of $y_{n+1} = y\left(x_n + h\right)$

$$y\left(x_n + h\right) = y\left(x_n\right) + h\, y'\left(x_n\right) + \frac{h^2}{2} y''\left(x_n\right) + O\left(h^3\right)$$

Modified Euler Method Contd……

Substitute $y'(x_n) = f(x_n, y_n)$

and

$$y'' = \frac{\partial}{\partial x} f(x_n, y_n) + \frac{d}{dx} y(x_n) \frac{\partial}{\partial y} f(x_n, y_n)$$

$$= \frac{\partial}{\partial x} f(x_n, y_n) + f(x_n, y_n) \frac{\partial}{\partial y} f(x_n, y_n)$$

to get

$$y(x_n + h) = y(x_n) + h f(x_n, y_n) + \frac{h^2}{2}\left[\frac{\partial}{\partial x} f(x_n, y_n) + f(x_n, y_n)\frac{\partial}{\partial y} f(x_n, y_n)\right] + O(h^3)$$

……….(5)

Modified Euler Method Contd……

From equation (4) and (5) it is clear that

$$y(x_{n+1}) = y(x_n) + \frac{1}{2}(k_1 + k_2) + O(h^3)$$

So the local truncation error is $O(h^3)$

The modified Euler method is second order accurate.

2^{nd} Order Runge-Kutta Method (RK2)

General 2^{nd} order Runge-Kutta method takes the form

$$k_1 = h\, f\left(x_n, y_n\right)$$
$$k_2 = h\, f\left(x_n + \alpha h, y_n + \beta k_1\right)$$
$$y_{n+1} = y_n + a_1 k_1 + a_2 k_2 \qquad \ldots\ldots\ldots\ldots(6)$$

Similar to the previous analysis

$$k_1 = h\, f\left(x_n, y\left(x_n\right)\right)$$ and

$$k_2 = h\, f\left(x_n, y\left(x_n\right)\right) + h^2\left[\alpha \frac{\partial}{\partial x} f\left(x_n, y\left(x_n\right)\right) + \beta\, f\left(x_n, y\left(x_n\right)\right)\frac{\partial}{\partial y} f\left(x_n, y\left(x_n\right)\right)\right] + O\left(h^3\right)$$

$$\therefore\ a_1 k_1 + a_2 k_2 = h\left(a_1 + a_2\right) f\left(x_n, y\left(x_n\right)\right)$$
$$+ a_2 h^2\left[\alpha \frac{\partial}{\partial x} f\left(x_n, y\left(x_n\right)\right) + \beta\, f\left(x_n, y\left(x_n\right)\right)\frac{\partial}{\partial y} f\left(x_n, y\left(x_n\right)\right)\right] + O\left(h^3\right)$$

2^{nd} Order Runge-Kutta Method (RK2) Contd….

Comparing

$$a_1 k_1 + a_2 k_2 = h\left(a_1 + a_2\right) f\left(x_n, y\left(x_n\right)\right)$$

$$+ a_2 h^2 \left[\alpha \frac{\partial}{\partial x} f\left(x_n, y\left(x_n\right)\right) + \beta\, f\left(x_n, y\left(x_n\right)\right) \frac{\partial}{\partial y} f\left(x_n, y\left(x_n\right)\right) \right] + O\left(h^3\right)$$

with equation (5)

$$y(x_n + h) = y\left(x_n\right) + h\, f(x_n, y_n) + \frac{h^2}{2}\left[\frac{\partial}{\partial x} f(x_n, y_n) + f(x_n, y_n) \frac{\partial}{\partial y} f(x_n, y_n) \right] + O\left(h^3\right)$$

……….(5)

To get a second order scheme $y_{n+1} = y_n + a_1 k_1 + a_2 k_2$

we must have $a_1 + a_2 = 1$ and

$$\alpha a_2 = \beta a_2 = \frac{1}{2}$$

2^{nd} Order Runge-Kutta Method (RK2) Contd….

So, general 2^{nd} order Runge-Kutta Scheme:

$$k_1 = h\,f\left(x_n, y_n\right)$$

$$k_2 = h\,f\left(x_n + \alpha h, y_n + \beta k_1\right)$$

$$y_{n+1} = y_n + a_1 k_1 + a_2 k_2$$

with

$$a_1 + a_2 = 1$$

$$\alpha a_2 = \beta a_2 = \frac{1}{2}$$

Since we have 3 equations and 4 unknowns a_1, a_2, α, β there are infinitely many solutions

2^{nd} Order Runge-Kutta Method (RK2) Contd….

The most popular are

- **Modified Euler method:**

$$a_1 = a_2 = \frac{1}{2}, \quad \alpha = \beta = 1$$

$$k_1 = h\, f\left(x_n, y_n\right), \quad k_2 = h\, f\left(x_n + h, y_n + k_1\right)$$

and

$$y_{n+1} = y_n + \frac{1}{2}\left(k_1 + k_2\right)$$

- **Midpoint Method:**

$$a_1 = 0, a_2 = 1, \quad \alpha = \beta = \frac{1}{2}$$

$$k_1 = h\, f\left(x_n, y_n\right), \quad k_2 = h\, f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

and

$$y_{n+1} = y_n + k_2$$

❑ Heun's Method

$$a_1 = \frac{1}{4}, a_2 = \frac{3}{4}, \quad \alpha = \beta = \frac{2}{3}$$

$$k_1 = h\,f\left(x_n, y_n\right), \quad k_2 = h\,f\left(x_n + \frac{2h}{3}, y_n + \frac{2k_1}{3}\right)$$

and

$$y_{n+1} = y_n + \frac{\left(k_1 + 3k_2\right)}{4}$$

4th Order Runge-Kutta Method (RK4)

Schemes of the form (6) can be extended to higher order methods. The most widely used Runge-Kutta scheme is the 4th order scheme RK4 based on Simpson's rule.

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

with

$$k_1 = hf\left(x_n, y_n\right)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_n + h, y_n + k_3\right)$$

This scheme has local truncation error of order $O\left(h^5\right)$, which can be checked in the same way as the second order scheme, but involves rather messy algebra.