Legendre Polynomials and Bessel Functions of first kind: Formula

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Legendre Equation:

The Legendre equation is given by

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0$$

$P_n\left(x\right)$ satisfy the above equation.

$P_n\left(x\right)$ is called Legendre Polynomial of order n. This $P_n\left(x\right)$ can be obtained from Rodrigue's Formula:

$$P_n\left(x\right)=\frac{1}{n!2^n}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

Few Legendre Polynomials are:

$$P_0\left(x\right)=1 \qquad P_1\left(x\right)=x$$

$$P_2\left(x\right)=\frac{3x^2-1}{2} \qquad P_3\left(x\right)=\frac{5x^3-3x}{2}$$

Orthogonality of Legendre Polynomials:

$$\int_{-1}^{1}P_m\left(x\right)P_n\left(x\right)dx=\begin{cases}0, & m\neq n\\ \dfrac{2}{2n+1}, & m=n\end{cases}$$

Recurrence Formulae for $P_n\left(x\right)$:

1. $\left(1-x^2\right)P_n'\left(x\right)=\left(n+1\right)xP_n\left(x\right)-\left(n+1\right)P_{n+1}\left(x\right)$

2. $\left(n+1\right)P_{n+1}\left(x\right)=\left(2n+1\right)xP_n\left(x\right)-nP_{n-1}\left(x\right)$

3. $nP_n\left(x\right)=xP_n'\left(x\right)-P_{n-1}'\left(x\right)$

4. $(2n+1)P_n(x) = P'_{(n+1)}(x) - P'_{(n-1)}(x)$

5. $P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$

Bessel's Equation

The most important differential equations is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0$$

$J_n(x)$ is a solution of this equation. This is called Bessel's equation of first kind.

Recurrence formulae for $J_n(x)$:

1. $nJ_n(x) + xJ'_n(x) = xJ_{n-1}$
2. $J'_n(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$
3. $J'_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$
4. $\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$
5. $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$

An important Property of Bessel functions:

$$J_{-n}(x) = (-1)^n J_n(x)$$

Orthogonalily of Bessel functions:

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \begin{cases} 0, & \alpha \neq \beta \\ \dfrac{\left[J_{n+1}(\alpha)\right]^2}{2}, & \alpha = \beta \end{cases}$$

where α and β are roots of $J_n(x) = 0$