Improper Integrals

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Consider the definite integral $\int_a^b f(x)\,dx$ which require to have

i) finite domain of integration $[a, b]$ and

ii) finite integrand i.e. $|f(x)| < \infty$.

The integral is said to be improper if a or b or both are infinite or the function $f(x)$ has infinite discontinuities .
So there are two types of improper integrals:

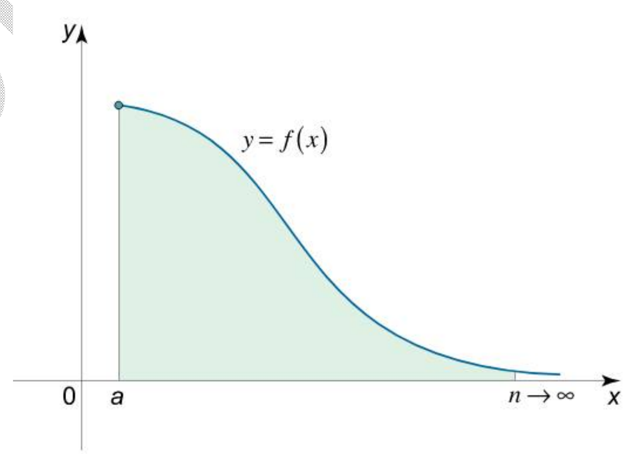
- **Improper integral of Type-1**: The limit a or b or both are infinite
- **Improper integral of Type-2**: The function has $f(x)$ one or more points of discontinuity in the interval $[a, b]$.

Type 1. Integration over an Infinite Domain

Let $f(x)$ be a continuous function on the interval [a,∞). We define the improper integral as

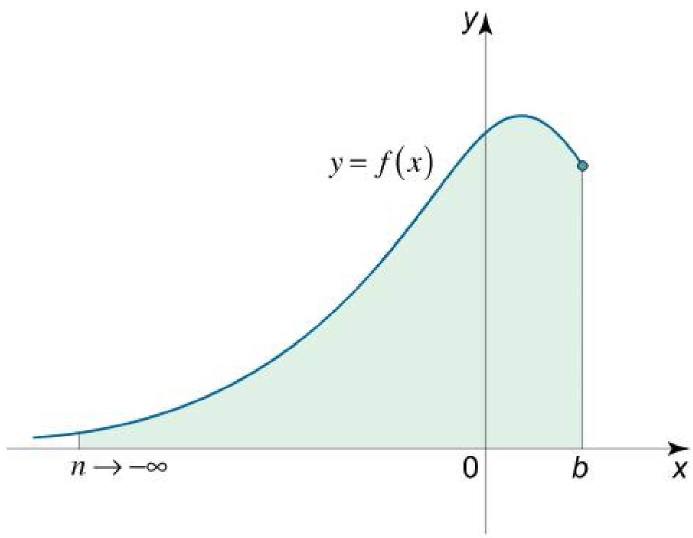
In order to integrate over the infinite domain [a,∞), we consider the limit of the form

$$\int_a^\infty f(x)\,dx = \lim_{r\to\infty}\int_a^r f(x)\,dx$$



Similarly, if a continuous function $f(x)$ is given on the interval $(-\infty, b]$, the improper integral of $f(x)$ is defined as

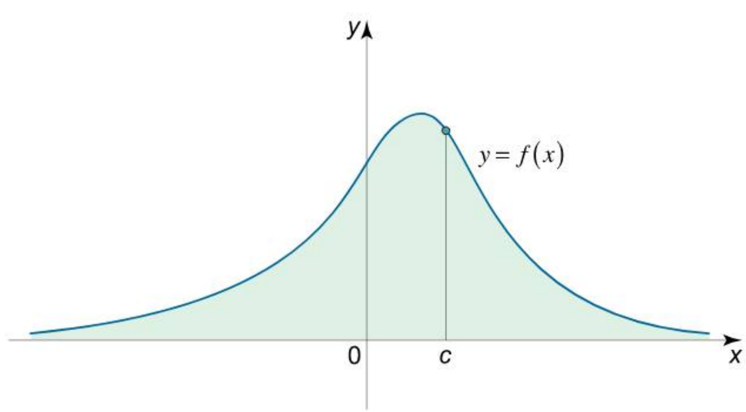
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{r\to-\infty}\int_{r}^{b} f(x)\,dx$$



If these limits exist and are finite then we say that the improper integrals are convergent. Otherwise the integrals are divergent.

An improper integral might have two infinite limits. In this case, we can pick an arbitrary point c and break the integral up there. As a result, we obtain two improper integrals, each with one infinite limit:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$



If, for some real number c, both of the integrals in the right-hand side are convergent, then we say that the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is also convergent; otherwise it is divergent.

Changing from infinte domains to finite domains of integration:

It is very much difficult to handle Improper integrals in numerical methods. One can't approximate an improper integral with infinite domain, because it is not known at what points the function is to be evaluated. If one encounter an integral with infinte domain, it is better to use a change of variable to change it to one with finite domain before applying numerical methods.

Consider the following integral $\int_{-\infty}^{\infty} f(x)\,dx$. We break the integral into three parts, a proper integral and two improper integrals with infinite domain.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

Substitute $x = \frac{1}{t}$ in the two improper integrals and after re-arranging we get

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} \left(f(x) + \frac{f\left(\frac{1}{x}\right)}{x^2} \right) dx$$

Note that the resulting integral is proper, since the integrand does not go to infinity at either end of the domain of integration.

Examples: **Examples of an improper integral on an infinite domain is**:

i) Gaussian integral $\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$

Other integrals

ii) $\int_{1}^{\infty} \frac{dt}{t^2}$ ii) $\int_{0}^{\infty} \frac{\sin(x)}{x}\,dx$ iii) $\int_{0}^{\infty} \frac{e^{-x}}{\sqrt{x}}\,dx$ iv) $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

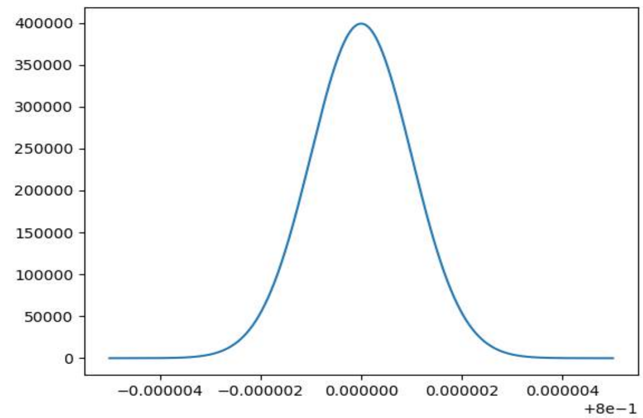
Numerically verifying the Gaussian integral result

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

Python Code

- Gaussian integral $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 1$

```
# Improper integral over infinite intervals
# Gaussian function: 1/(sigma*sqrt(2*pi))*exp(-(x-
mu)**2/(2*sigma**2))
# The results is 1

import numpy as np
from scipy.integrate import quad,simps
import matplotlib.pyplot as plt
# Parameter for Gaussian Function
sig=0.001
mu=0.8

# Definition of Gaussian function

f=lambda x: 1.0/(sig*np.sqrt(2*np.pi))*np.exp(-(x-
mu)**2/(2*sig**2))

x=np.linspace(-0.2,1.8,1000)



for i in range(-1,-7,-1):
         sig=10**i
         # Two different methods of evaluating the integral
         # for different values of sigma to look into details of
the problems
         I1=simps(f(x),x)
         I2=quad(f,mu-5*sig,mu+5*sig)[0]

         print sig, I1, I2

x=np.linspace(mu-5*sig,mu+5*sig,1000)
plt.plot(x,f(x))
plt.savefig("gaussian_imp.png")
plt.show()
```



Sigma	simps	quad(f, mu-5sig,mu+5sig)
0.1	1.0	0.999999426697
0.01	1.0000000000000002	0.999999426697
0.001	0.98547364558662	0.999999426697
0.0001	2.7873133470889672e-21	0.999999426697
1e-05	0.0	0.999999426701
1e-06	0.0	0.999999426702

- **Using change of variables method**

```
# Improper integral over infinite intervals
# Gaussian function: 1/(sigma*sqrt(2*pi))*exp(-(x-
mu)**2/(2*sigma**2))
# The results is 1
# Using change of variables g(x)=f(x)+f(1/x)/x**2 with limit -1 to
+1

import numpy as np
from scipy.integrate import quad,simps
import matplotlib.pyplot as plt

# Parameter for Gaussian Function
sig=0.001
mu=0.8

# Definition of Gaussian function

f=lambda x: 1.0/(sig*np.sqrt(2*np.pi))*np.exp(-(x-
mu)**2/(2*sig**2))

g=lambda x: f(x)+f(1.0/x)/x**2

x=np.linspace(-1,1,999)
```

```
for i in range(-1,-7,-1):
         sig=10**i
         I=simps(g(x),x)
         I2=quad(g,-1,1,points=0)[0]
         I3=quad(g,mu-5*sig,mu+5*sig)[0]

         print sig,I, I2, I3
```

Results:

Sigma	simps	quad(g,-1,1)	quad(g,mu-5sig, mu+5sig)
0.1	1.0000000881064954	1.0	1.62084118993
0.01	1.0000000000000002	1.0	0.999999426697
0.001	1.011752073809608	1.24173561522e-74	0.999999426697
0.0001	1.0765787971410394	0.0	0.999999426697
1e-05	7.404563728745796e-86	0.0	0.999999426701
1e-06	0.0	0.0	0.999999426702

Note: It should be noted from the above results that as the width (σ)of the Gaussian Function decreases the simps() and quad() methods for the limit -1 to +1 do not give correct value after a certain value of σ. This is because that when sigma is very small the Gaussian function gets too narrow and the region of integration becomes smaller . The main contribution of the integration comes from the central small part of the curve. So instead of taking large limits (-infinity to + infinity) we can concentrate on the central portion from where the main contribution comes. So, what will be the appropriate limit of integration for this Gaussian function? Since the σ is the standard deviation which indicates the scale, we can take limit which extends between a few σ distance on either side of the central peak. For example we can take -5σ to 5σ. We can check the results by varying these limits (increasing or decreasing).

- $\int_0^{\infty} \frac{\sin(x)}{x} dx$

```
# Improper integral over infinite intervals
# Gaussian function: sin(x)/x  limit 0 to inf
# The results is pi/2


import numpy as np
from scipy.integrate import quad,simps
import matplotlib.pyplot as plt
```

```
# Definition of function

f=lambda x: np.sin(x)/x

x=np.arange(0.001,100,0.0001)


I1=simps(f(x),x)

# Printing results
print I1

# Plotting function
plt.plot(x,f(x))
plt.show()



# Result:  1.5612259733536067
```

- **Numerically verifying the Gaussian integral result**

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

```
# Numerically verifying Gaussian integral results
# Integration(exp(-ax**2+b*x+c), with limit -inf to inf)
# Exact result sqrt(pi/a)*exp(b**2/(4*a)+c)

import numpy as np
from scipy.integrate import quad,simps
import matplotlib.pyplot as plt

# Parameter of the given Function
a,b,c=input("Enter a,b,c \n")
l,h=input("Enter lower and upper limit of integration\n")
dx=0.001
inc=2
# Definition of function
f=lambda x: np.exp(-a*x**2+b*x+c)
x=np.arange(l,h,dx)

# Evaluation of integration
diff=10
I0=simps(f(x),x)
while abs(diff)>0.001:
         l=l-inc
         h=h+inc
```

```
         x=np.arange(l,h,dx)
         I=simps(f(x),x)
         diff=abs(I-I0)
         I0=I
# Exact result (RHS)

exact=np.sqrt(np.pi/a)*np.exp(b**2/(4.0*a)+c)

print exact,I
plt.plot(x,f(x))

plt.show()
```

Results:

```
Enter a,b,c
7,4,1
Enter lower and upper limit of integration
-1,1
3.224695218291411    3.2246952182914113
```